Ministry of Higher Education & Scientific Research
Al-Rasheed University College
Department of Medical Instrumentation Engineering

وزارة التعليم العالي والبحث العلمي
كلية الرشيد الجامعة
قسم تقنيات الاجهزة الطبية

حقيبة تعليمية

بعنوان: رياضيات 2

اعداد :

اعد اد
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التدريسي الثانوي: م.د نجم عبدالله جازع

2022-2023

المقدمة

يتسم البرنامج التعليمي الرياضيات 2 بالتدريس باللغة الانكليزية لمدة ثلاثون اسبوعا بواقع ثلاث ساعات نظري اسبوعيا يتم تدريس الطلبة القوانين والمسائل الرياضية اللازمة لغرض حل الدوائر الكهربائية البسيطة والمعقدة ضمن منهج متكامل ، و الاطلاع على الاعداد المركبة والاعداد التخيلية وفهم ومعرفة التطبيقات العملية لقوانين والمسائل الرياضية،التعرف على المعادلات الرياضية الخاصة بالتفاضل وكيفية حلها، شرح نظرية التكامل من خلال مفهوم المساحة، فهم ومعرفة المعادلات الرياضية اللازمة والتطبيقات للمصفوفات.

وصف المقرر الدراسي

يوفر وصف المقرر هذا إيجازاً مقتضياً لأهم خصائص المقرر ومخرجات التعلم المتوقعة من الطالب تحقيقها مبرهناً عما إذا كان قد حقق الاستفادة القصوى من فرص التعلم المتاحة. ولابد من الربط بينها وبين وصف البرنامج.؛

17. المؤسسة التعليمية	كلية الرشيد الجامعة
18. القسم العلمي / المركز	هندسة تقنيات الاجهزة الطبية
19. اسم / رمز المقرر	رياضيات
20. أشكال الحضور المتاحة	اسبوعي (نظري)
21. الفصل / السنة	سنوي
22. عدد الساعات الدراسية (الكلي)	90 ساعة
23. تاريخ إعداد هذا الوصف	2021-6-28
24. أهداف المقرر	
1- مساعدة الطالب على تفهم القوانين والمسائل الرياضيه اللازمه لغرض حل الدوائر الكهربائيه البسيطه والمعقده	
2-الارتقاء بمستوى الطالب في ماده الرياضيات	
3-تنميه اساليب تفكير سليميه واطلاق الطاقات الكامنه عند الطالب وتطبيقها في المجال الهندسي	

14. مخرجات المقرر وطرائق التعليم والتعلم والتقييم
أ- الأهداف المعرفية أ1-ان يذكر الطالب مثلا(نص مبرهنه كرين- تعريف المتجه......) أ2- ان يميز الطالب بين الضرب النقطي والضرب الاتجاهي أ3- ان يستخدم الطالب اكثر من طريقه لحل المعادلات التفاضليه أ4- ان يتعرف الطالب على انواع الاحداثيات أ5- ان يفهم الطالب كيفيه ايجاد التكامل المتكرر أ6- ان يحكم الطالب على صحه الاستنتاجات التي يصل اليها
ب - الأهداف المهاراتية الخاصة بالمقرر. ب1 -حل بعض المشكلات الرياضيه وحل اسئله غير نمطيه تتطلب مهارات متعدده ب2 – الدقه والوضوح والانجاز في التعبير ب3 – تنميه القدرات على التفكير المنطقي المتسلسل ب4- صياغه مشكله حياتيه صياغه رياضيه واستخدام اساليب رياضيه في حلها
طرائق التعليم والتعلم
محاضرات –وسائل الايضاح(data show)
طرائق التقييم
امتحانات فصلية تحريرية اختبارات اسبوعية/ شفوية + تحريرية اسئلة سريعة اسئلة قبلية وبعدية

ج- الأهداف الوجدانية والقيمية ج1-ان يصغي الطالب بانتباه الى شرح الاستاذ ج2-ان يهتم الطالب بهدوء وتظام الصف ج3-ان يتعرف الطالب على اثر العلم والعلماء في الحياة ج4- ان يصف الطالب اهميه تعلم الرياضيات مثلا
طرائق التعليم والتعلم
المناقشة والحوار مع الطلبة
طرائق التقييم
استبيان، ندوات، محاور نقاش
د - المهارات العامة والتأهيلية المنقولة (المهارات الأخرى المتعلقة بقابلية التوظيف والتطور الشخصي). د1-اكتساب الخريج مهارات تاسيسيه لماده الرياضيات من حيث اللغه والرموزوالمعلومات واساليب التفكير د2-تنميه مهارات عقليه تمكن الخريج من الاستفاده من المعلومات التي يتعلمها والمهارات التي اكتسبها وتوظيفها في خدمه متطلباته كفرد وفي خدمه اهداف المجتمع من حيث التنميه الاجتماعيه والاقتصاديه د3-اكتساب بعض المهارات العمليه مثل استخدام الادوات الهندسيه ومهارات القياس وتشغيل بعض الاجهزة والالات د4- تنميه اساليب تفكيرسليمه واطلاق الطاقات الكامنه

15. بنية المقرر

الأسبوع	الساعات	مخرجات التعلم المطلوبة	اسم الوحدة / أو الموضوع	طريقة التعليم	طريقة التقييم
1	3	الطالب يفهم الدرس	Vector analysis	محاضره نظري	امتحانات اسبوعيه- اسئله قبليه وبعديه
2	3	الطالب يفهم الدرس	Vector field	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه
3	3	الطالب يفهم الدرس	Linear algebra	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه
4	3	الطالب يفهم الدرس	Vector calculus	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه
5	3	الطالب يفهم الدرس	Scalars and vector unit	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه
6	3	الطالب يفهم الدرس	Orthogonal vector	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه
7	3	الطالب يفهم الدرس	Dot product	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه
8	3	الطالب يفهم الدرس	cross product	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه
9	3	الطالب يفهم الدرس	Theory for vector field	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه
10	3	الطالب يفهم الدرس	Vector variable function	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه
11	3	الطالب يفهم الدرس	Polar coordinates – gradient in polar	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه

امتحانات اسبوعيه اسئله قبليه وبعديه	محاضره نظري	Spherical coordinates	الطالب يفهم الدرس	3	12
امتحانات اسبوعيه اسئله قبليه وبعديه	محاضره نظري	Complex number	الطالب يفهم الدرس	3	13
امتحانات اسبوعيه اسئله قبليه وبعديه	محاضره نظري	Polar form of complex number	الطالب يفهم الدرس	3	14
امتحانات اسبوعيه اسئله قبليه وبعديه	محاضره نظري	Algebra for complex number	الطالب يفهم الدرس	3	15
امتحانات اسبوعيه اسئله قبليه وبعديه	محاضره نظري	Algebra for Spherical coordinates	الطالب يفهم الدرس	3	16
امتحانات اسبوعيه اسئله قبليه وبعديه	محاضره نظري	Infinite series	الطالب يفهم الدرس	3	17
امتحانات اسبوعيه اسئله قبليه وبعديه	محاضره نظري	Power series	الطالب يفهم الدرس	3	18
امتحانات اسبوعيه اسئله قبليه وبعديه	محاضره نظري	Convergence and divergence series	الطالب يفهم الدرس	3	19
امتحانات اسبوعيه اسئله قبليه وبعديه	محاضره نظري	Number and Complex series	الطالب يفهم الدرس	3	20
امتحانات اسبوعيه اسئله قبليه وبعديه	محاضره نظري	Complex variable	الطالب يفهم الدرس	3	21
امتحانات اسبوعيه اسئله قبليه وبعديه	محاضره نظري	Cauchy - Riemann equations	الطالب يفهم الدرس	3	22
امتحانات اسبوعيه	محاضره نظري	Differential equation	الطالب يفهم الدرس	3	23

					اسئله قبليه وبعديه
24	3	الطالب يفهم الدرس	Differential equation of the first order	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه
25	3	الطالب يفهم الدرس	Differential equation of n order	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه
26	3	الطالب يفهم الدرس	Application	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه
27	3	الطالب يفهم الدرس	Multiple integrations	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه
28	3	الطالب يفهم الدرس	Surface area	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه
29	3	الطالب يفهم الدرس	Green theorem	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه
30	3	الطالب يفهم الدرس	Stokes theorem	محاضره نظري	امتحانات اسبوعيه اسئله قبليه وبعديه

1. البنية التحتية	
تب المقررة المطلوبة	Calculus II
راجع الرئيسية (المصادر)	الكتب – الانترنت
ب والمراجع التي يوصى بها) ت العلمية ، التقارير ،....)	Calculus Thomas -13th edition Schaum's mathematic book Practice problem calculus II Topic s in a calculus II-wolfram mathworld
مراجع الالكترونية، مواقع الانترنيت	

11. خطة تطوير المقرر الدراسي
1-اضافه تحويلات لابلاس لاستفادة منه في الدروس الهندسيه 2- استخدام اللغه البرمجيه في الرياضيات التطبيقيه

X

فهرس المحتويات

إرشادات للطلبة

- الرغبة والحماس للتعليم
- كن مشاركاً في جميع الأنشطة
- احترم أفكار المدرس والزملاء
- أنقد أفكار المدرس والزملاء بأدب إن كانت هناك حاجة.
- احرص على استثمار الوقت
- تقبل الدور الذي يسند إليك في المجموعة
- حفز أفراد مجموعتك في المشاركة في النشاطات
- احرص على بناء علاقات طيبة مع المدرس والزملاء أثناء المحاضرة
- احرص على ما تعلمته في المحاضرة وطبقه في الميدان .
- ركز ذهنك بالتعليم واحرص على التطبيق المباشر
- تغلق الموبايل قبل الشروع بالمحاضرة

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand *Vector analysis*

c. Central Ideas:

- **Introduction**
- **Vector Arithmetic**
- **Scalars and vector unit**

Objectives: after the end of courses the student will be able to:

- Define **Scalar Quantities , Vector Quantities , Unit vector and magintud vector**
- **Find addition of the vector, subtraction , scalar multiplication**

2. Pre test:

Q1-fill in the blanks within an appropriate word(s):

1- The vector $\overrightarrow{AB}$ from A=(2,-7,0) to B= (1,-3,-5) is …………

2- The length of the vector , $\vec{v}$=(-1 , 3 , 2) is …………

Q2- Find the vector $\overrightarrow{A}$ directed from point (2,-4,1) to point(0,-2,0) in Cartesian coordinates and find the unit vector along $\overrightarrow{A}$

Q3-

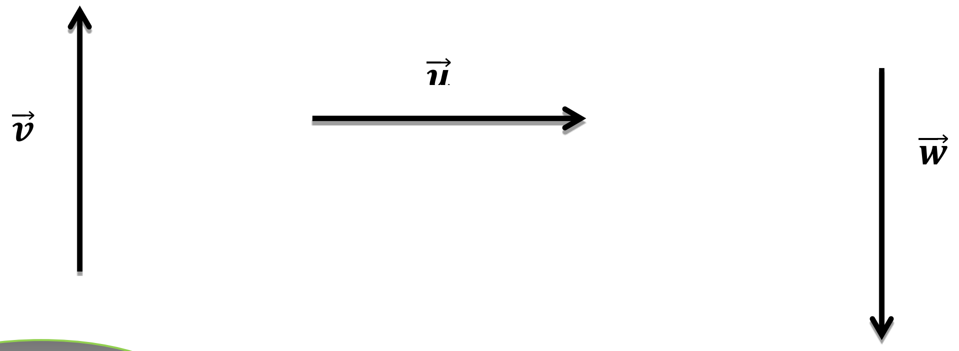
1) Find $\vec{u} + \vec{v}$ for $\vec{u}$ = (3i + 4j), $\vec{v}$= (-2i + j).

2) Find $\vec{u} - \vec{v}$ for $\vec{u}$= 5(3i + 2j), $\vec{v}$ = (7i + 3j).

Definition **Scalar Quantities** A scalar is a quantity which has only a magnitude in space. Such as: length, weight, volume, Temperature… etc..

Definition **Vector Quantities** are a quantity which has both Magnitude and direction in space such as: force, speed….. etc

$\vec{v}=\overrightarrow{AB}$, $\vec{u}=\overrightarrow{CD}$, $\vec{w}=\overrightarrow{EF}$



Note

(1) The zero vector is just a point, and it is denoted by 0, and has arbitrary direction ,which is written as $\vec{0}$

(2) Given the two points $A=(a_0,b_0,c_0)$ and $B=(a_1,b_1,c_1)$ the vector with the representation $\overrightarrow{AB}=(a_1-a_0,b_1-b_0,c_1-c_0)$

Note that the vector above is the vector that starts at A and ends at B.

The vector that starts at B and ends at A is $\overrightarrow{BA}=(a_0-a_1,\ b_0-b_1,c_0-c_1)$

(3) $\overrightarrow{AB}=-\overrightarrow{BA}$

Example 1: Given the vector for each of the following.

1- The vector from A=(2,-7,0) to B= (1,-3,-5)
2- The vector from C=(1,-3,-5) to D= (2,-7,0)

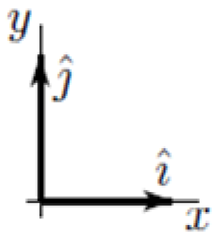
Solution

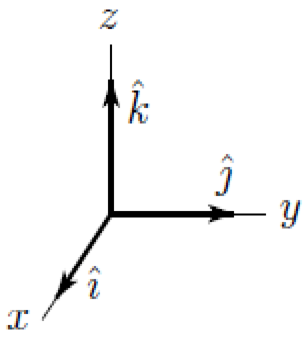
1- $\overrightarrow{AB}=(1-2,-3+7,-5-0)=(-1,4,-5)$
2- $\overrightarrow{CD}=(2-1,-7+3,0+5)=(1,4,5)$

Definition **Unit vector** A vector of length 1 is called a *unit vector*. In an *xy*-coordinate system the unit vectors along the *x*- and *y*-axis are denoted by i and j, respectively. In an *xyz*-coordinate system the unit vectors along the *x*-, *y*- and *z*-axis are denoted by i, j and k, respectively. Thus:

$\vec{i}$ = (1,0) , j = (0,1) (2- dimension)

$\vec{i}$ = (1,0,0) , j = (0,1,0), $\vec{k}$ = (0,0,1) (3- dimension)





Note

All vectors can be expressed as linear combinations of the unit vectors

$\vec{v}$ = (v_1 , v_2)= v_1i + v_2j (2- dimension)

$\vec{v}$ = (v_1 , v_2 , v_3)= v_1i + v_2j + v_3k (3- dimension)

Example Write the vector $\vec{u}$ = (3, 4,5) as linear combinations of the unit vector

Solution

$\vec{u}$ = 3i + 4j + 5k

Definition magnitude or (length) The magnitude or (length) of the vector $v = (v_1, v_2, v_3)$ is denoted by the symbol$||\vec{v}||$ or $|\vec{v}|$ is.

$$||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Properties of the magnitude

a) $\vec{v} \cdot \vec{v} = || \vec{v}||^2$

b) $|| \vec{0}|| = 0$

c) $|| \vec{v}|| \geq 0$

d) $|| \vec{v}|| = 0$ if and only if $\vec{v} = \vec{0}$

e) $|| \vec{v} + \vec{u}|| \leq || \vec{v}|| + || \vec{u}||$

f) $|| \vec{v} . \vec{u}|| \leq || \vec{v}|| . || \vec{u}||$

Example :

Find the length of the vector , $\vec{v}$=(-1 , 3 , 2)

Solution

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-1)^2 + (3)^2 + (2)^2} = \sqrt{1+9+4} = \sqrt{14}$$

Note

The magnitude of a vector is *not* in general equal to the sum of the magnitudes of the two original vectors.

For Example

The magnitude of the vector (3, 0, 0) is 3, and the magnitude of the vector (-2, 0, 0) is 2,
but the magnitude of the vector(3, 0, 0) + (-2, 0, 0) is 1, not 5!

A unit vector (u_n) for any vector $\vec{A}$

Defined as a vector whose magnitude is unity and is along of $\vec{A}$ that is

$$u_n = \frac{\vec{A}}{|\vec{A}|}$$

Example:

Find the vector $\vec{A}$ directed from point (2,-4,1) to point(0,-2,0) in Cartesian coordinates and find the unit vector along $\vec{A}$

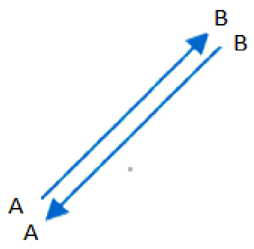
Solution

$$\vec{A} = -2i+2j-k$$

$$|\vec{A}| = \sqrt{(-2)^2 + (2)^2 + (-1)^2} = 3$$

$$u_n = \frac{\vec{A}}{|\vec{A}|} = \frac{-2i+2j-k}{3} = \frac{-2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$

Definition **Inverse Vectors** **An inverse vector is a vector of equal magnitude to the original but in the opposite direction**



- $\overrightarrow{AB} = -\overrightarrow{BA}$
- $\overrightarrow{AB} + \overrightarrow{BA} = 0$

Vector Arithmetic

1- addition of the vector:

Given the two vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$
The addition of the two vectors is given by the following formula

$$\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$$

For $\vec{v} = a_1 i + a_2 j + a_3 k$ and $\vec{w} = b_1 i + b_2 j + b_3 k$ be two vectors
Then $\vec{v} + \vec{w} = (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k$

2- Subtraction of vector:

Given the two vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{u} = (u_1, u_2, u_3)$
the Subtraction vector is

$$\vec{v} - \vec{u} = (v_1 - u_1, v_2 - u_2, v_3 - u_3)$$

For $\vec{v} = a_1 i + a_2 j + a_3 k$ and $\vec{w} = b_1 i + b_2 j + b_3 k$ be two vectors
Then $\vec{v} - \vec{w} = (a_1 - b_1)i + (a_2 - b_2)j + (a_3 - b_3)k$
The subtraction operation between two vectors $\vec{u} - \vec{v}$ can be understood as a vector addition between the first vector and the opposite of the second vector:

Definition scalar multiplication

Given the vectors $\vec{v} = (v_1, v_2, v_3)$ and any number C scalar multiplication is

$$C\,\vec{v} = (Cv_1, Cv_2, Cv_3)$$

For $\vec{v} = v_1 i + v_2 j + v_3 k$

Then $C\,\vec{v} = Cv_1 i + Cv_2 j + Cv_3 k$

Scalar multiplication obeys the following rules :

- Distributive in the scalar: $(k + d)\vec{v} = k\vec{v} + d\vec{v}$
- Distributive in the vector: $k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}$
- Associate of product of scalars with scalar multiplication: $(kd)\vec{v} = k(d\vec{v})$
- Multiplying by 1 does not change a vector: $1\vec{v} = \vec{v}$
- Multiplying by 0 gives the zero vector: $0\vec{v} = \vec{0}$
- Multiplying by −1 gives the additive inverse: $(-1)\vec{v} = -\vec{v}$

Properties of vector algebra

If $\vec{v}$, $\vec{w}$ and $\vec{u}$ are vectors and a and b two numbers then we have the following properties:

a. For any vector v there is a vector (−v) such that $\vec{v} + (-\vec{v}) = \vec{0}$

b. $\vec{v}+\vec{w}=\vec{w}+\vec{v}$ Commutative Law

c. $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ Associative Law

d. $\vec{v}+0 = \vec{v} = 0+\vec{v}$ Additive Identity

e. $(a + b)\vec{v} = a\vec{v} + b\vec{v}$

Example

3) Find $\vec{u} + \vec{v}$ for $\vec{u}$ = (3i + 4j), $\vec{v}$= (-2i + j).

4) Find $\vec{u} - \vec{v}$ for $\vec{u}$= 5(3i + 2j), $\vec{v}$ = (7i + 3j).

Solution

1) $\vec{u} + \vec{v}$ = (3i + 4j) + (-2i + j) = (3 - 2)i + (4 + 1)j = 1i + 5j = i + 5j

2) 5 (3i + 2j) - (7i + 3j) = (15i + 10j) - (7i + 3j) = (15 - 7)i + (10 - 3)j = 8i + 7j

Note

Let $\vec{v} = a_1 i + a_2 j + a_3 k$ and $\vec{w} = b_1 i + b_2 j + b_3 k$ be two vectors and

$$\vec{v} = \vec{w} \Leftrightarrow a_1 = b_1 \, and \;\; a_2 = b_2 \, and \; a_3 = b_3$$

Example: Consider the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in R^3,
where $P = (2,1,5), Q = (3,5,7)$,
$R = (1,-3,-2)$ and $S = (2,1,0)$. Does $\overrightarrow{PQ} = \overrightarrow{RS}$?

Solution

$\overrightarrow{PQ}$= Q-P=(3−2,5−1,7−5) = (1,4,2).

$\overrightarrow{RS}$= (2−1,1−(−3),0−(−2)) = (1,4,2).

$\therefore \overrightarrow{PQ} = \overrightarrow{RS} = (1,4,2).$

Example:

Door $\vec{u} = (-2, 1)$, $\vec{v} = (-2, 1)$

1) for $\vec{u} = (5, 3)$, $\vec{v} = (3, 5)$

Solution

1) $\vec{u} = \vec{v}$

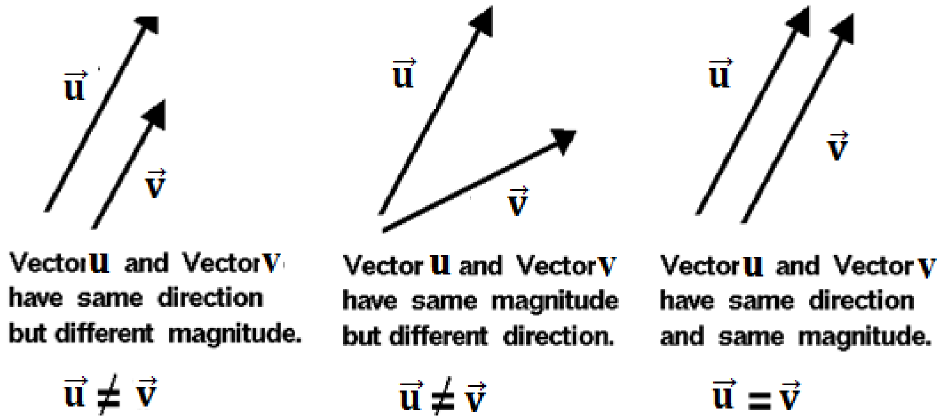
because $u_1 = -2, v_1 = -2 \Rightarrow u_1 = v_1$ and $u_2 = 1, v_2 = 1 \Rightarrow u_2 = v_2$

2) $\vec{u} \neq \vec{v}$

because. $u_1 = 5$ $v_1 = 3$, $\Rightarrow u_1 \neq v_1$, $u_2 = 3,$ $v_2 = 5, \Rightarrow u_2 \neq v_2$

Note

Two nonzero vectors are equal if they have the same magnitude and the same direction. Any vector with zero magnitude is equal to the zero vector.



1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand vector **analysis**

c. Central Ideas:

- **Dot pordect**
- **Properties of dot prodect**
- **Projection**
- **unit vector normal**

d. Objectives: after the end of courses the student will be able to:

Find

- **Dot product**
- **Angle between the vector**
- **projection**
- **Normal unit vector**

2. Pre test: fill in the blanks within an appropriate word(s):

1-Two non zero vectors $\vec{u}$ and $\vec{v}$ are said to parallel if …………

2- Two non zero vectors $\vec{u}$ and $\vec{v}$ are said to ………… if $\vec{u}.\vec{v} = 0$

3- Given $\vec{A} = 3i - 2j + k$ and $\vec{B} = mi + j - 3k$
the constant (m)= ……….. if the vectors $\vec{A}$ and $\vec{B}$ are orthogonal

3-Dot Product

Definition Let $\vec{v} = (v1,v2,v3)$ and $\vec{w} = (w1,w2,w3)$ be vectors in R^3.
The dot product of $\vec{v}$ and $\vec{w}$, denoted by $\vec{v}\cdot\vec{w}$, is given by:

$$\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + v_3w_3$$

Similarly, for vectors $v = (v_1,v_2)$ and $w = (w_1,w_2)$ in R^2, thot product is:

$$\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2$$

For vectors $v = v_1\,i + v_2\,j + v_3k$ and $w = w_1\,i + w_2\,j + w_3k$ in component form, the dot product is still

$$\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + v_3w_3$$

Properties Of The Dot Product

a) $\vec{u}\,.\,\vec{v} = \vec{v}.\vec{u}$ (commutative)
b) $(s\vec{u}).\vec{v} = s(\vec{u} \cdot \vec{v})$ (respects scalar multiples)
c) $\vec{u}\,.(\vec{v} + \vec{w}) = \vec{u}\,.\,\vec{v} + \vec{u}\,.\,\vec{w}$ (distributes over vector sums)
d) $\vec{0} \cdot \vec{u} = 0$
e) $\vec{u}\,.\,\vec{v} = 0 \Leftrightarrow \vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$or $\vec{u} \perp \vec{v}$
f) $s\vec{u}.k\vec{v} = s.k(\vec{u}.\vec{v})$

Note

The associative law does *not* hold for the dot product of vectors Because for vectors u, v, w, the dot product u·v is a scalar, and so (u·v) ·w is not defined since the left side of that dot product (the part in parentheses) is a scalar and not a vector.

Example

given $\vec{u}=(2\,,-2\,)$, $\vec{v}= (5, 8)$, $\vec{w} =(-4, 3\,)$ find each of the following:
1) $\vec{u}\,.\,\vec{v}$
2) $(\vec{u}\,.\,\vec{v})\,.\,\vec{w}$
3) $\vec{u}\,.(\,2\vec{v}\,)$

Solution:

1) $\vec{u}\,.\,\vec{v} =(2\,,-2\,)\ .\,(5\ ,8) = 2x\,5 + (-2)x8= 10-16= -6$
2) $(\vec{u}\,.\,\vec{v})\,.\,\vec{w} = -6\,(-4, 3\,) = (-6\ x\text{-}4\,,\ -6\ x\ 3)$
3) $\vec{u}\,.(\,2\vec{v}\,) = 2\,(\vec{u}\,.\,\vec{v}) = 2x\,\text{-}6 =-12$

The Dot Product Of i, j And k

i · i = 1 , j · j = 1 , k · k = 1

i · j = 0 , j · i = 0 , k · i = 0

i · k = 0 , j · k = 0 , k · j = 0

Define Projection let be $\vec{A}$ and $\vec{B}$ are vectors the projection of $\vec{B}$ onto $\vec{A}$ $proj_A\vec{B}$

Is given by

$$proj_A\vec{B} = \frac{\vec{A}.\vec{B}}{|\vec{A}|^2}\vec{A}$$

Note

The projection of $\vec{A}$ onto $\vec{B}$ $proj_A\vec{B}$ is given by

$$proj_B\vec{A} = \frac{\vec{A}.\vec{B}}{|\vec{B}|^2}\vec{B}$$

Example:

Determine the projection of vector $\vec{B}$=(2,1,-1)onto vector $\vec{A}$ =(1,0,-2)

Solution

$$proj_A\vec{B} = \frac{\vec{A}.\vec{B}}{|\vec{A}|^2}\vec{A}$$

$\vec{A}.\vec{B} = 4$

$|\vec{A}| = \sqrt{(1)^2 + (0)^2 + (-2)^2} = \sqrt{5}$

$$proj_A\vec{B} = \frac{4}{5}(i - 2k)$$

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand Theory for vector

c. Central Ideas:

- **(Direction cosines)**
- **Direction angle**
- **Cross Product**
- **unit vector normal**

d. Objectives: after the end of courses the student will be able to:

Find

- **Direction cosines , Direction angle**
- **Angle between the vector**
- **Normal unit vector**

2. Pre test: fill in the blanks within an appropriate word(s):

Q1- Determine the direction cosines and direction angle for $\vec{v} = (2, 1, -4)$

Q2- Fill in the following blanks

1- If $\vec{u}$ = i – 2j + k and $\vec{v}$ = 3i + j – 2k then $\vec{v}$ x $\vec{u}$ …………

2- Two non zero vectors $\vec{u}$ and $\vec{v}$ are said to ………… if $\vec{u}.\vec{v} = 0$

3- Given $\vec{A} = 3i - 2j + k$ and $\vec{B} = mi + j - 3k$ the constant (m)= ……….. if the vectors $\vec{A}$ and $\vec{B}$ are orthogonal

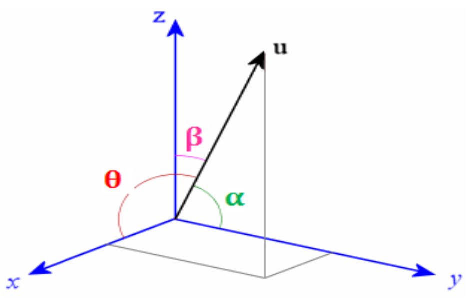
(Direction cosines)

This application direction of the dot product requires that we be in three dimensional spaces unlike all the other application we have looked at to this point

Defection **let $\vec{v}$ =(v_1,v_2 , v_3)be a vector three dimensional space and θ is the angle a vector makes with the x axis, α is the angle a vector makes with the y axis, and β is the angle a vector makes with the z axis. These angles are called** **Direction angle** **and the cosines of these angle are called** **Direction cosines**

The formulas for the direction cosines are

$$\cos\theta = \frac{v_1}{\|v\|} \Rightarrow v_1 = \|v\| \cos\theta$$
$$\cos\alpha = \frac{v_2}{\|v\|} \Rightarrow v_2 = \|v\| \cos\alpha$$
$$\cos\beta = \frac{v_3}{\|v\|} \Rightarrow v_3 = \|v\| \cos\beta$$



Note

For any vector $\vec{v}$ in Cartesian three-space, the sum of the squares of the direction cosines is always equal to 1.

$$\cos^2\theta + \cos^2\alpha + \cos^2\beta = 1$$

Example

If ||v|| = 5 and θ=70°, α=85°,β = 20° give the component form vector $\vec{v}$.

Solution

$\vec{v}$ = (||v|| cos θ , ||v|| cos α, ||v|| cos β)

= (5 cos70 , 5 cos85, 5 cos20)

Example

Determine the direction cosines and direction angle for $\vec{v} = (2, 1, -4)$

Solution

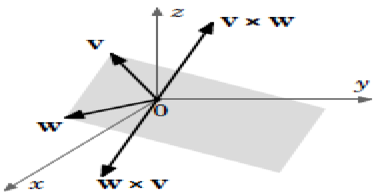
$$\|\vec{v}\| = \sqrt{4+1+16} = \sqrt{21}$$
$$\cos\theta = \frac{2}{\sqrt{21}} \Rightarrow \theta = 64.123$$
$$\cos\alpha = \frac{1}{\sqrt{21}} \Rightarrow \alpha = 77.396$$
$$\cos\beta = \frac{-4}{\sqrt{21}} \Rightarrow \beta = 150.794$$

4-Cross Product:

Definition let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be vectors in space , the cross product of $\vec{u}$ and $\vec{v}$ is the vector :

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2 \;,\; u_3v_1 - u_1v_3 \;,\; u_1v_2 - u_2v_1)$$

The cross product $\vec{u}\times\vec{v}$ of two nonzero vectors $\vec{u}$ and $\vec{v}$ is also a nonzero vector, it is perpendicular to both $\vec{u}$ and $\vec{v}$.



We can now rewrite the definition for the cross product using these determinants:

a) **The top row consists of the unit vectors in order $\vec{i}, \vec{j}, \vec{k}$.**
b) **The second row consists of the coefficients $\vec{u}$.**

c) **The third row consists of the coefficients $\vec{v}$.**

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$$

← Put "**u**" in Row 2.

← Put "**v**" in Row 3.

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}\mathbf{k}$$

$$=\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\mathbf{k}$$

$$=(u_2v_3-u_3v_2)\mathbf{i}-(u_1v_3-u_3v_1)\mathbf{j}+(u_1v_2-u_2v_1)\mathbf{k}$$

Properties of the cross product:

If $\vec{v}$, $\vec{u}$, and $\vec{w}$ are vectors and s is a scalar, then :

1. $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$ **(Anti-commutative)**
2. $(s\,\vec{v}) \times \vec{u} = s(\vec{v} \times \vec{u}) = \vec{v} \times (s\vec{u})$
3. $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$ **(Distributive)**
4. $(\vec{v} + \vec{u}) \times \vec{w} = \vec{v} \times \vec{w} + \vec{u} \times \vec{w}$
5. $\vec{v} \cdot (\vec{u} \times \vec{w}) = (\vec{v} \times \vec{u}) \cdot \vec{w} = \vec{u}.\,(\vec{w} \times \vec{v})$
6. $\vec{v} \times (\vec{u} \times \vec{w}) \neq (\vec{v} \times \vec{u}) \times \vec{w}$ **(Not associative)**
7. $\vec{v} \times (\vec{u} \times \vec{w}) = (\vec{v} \cdot \vec{w})\,\vec{u} - (\vec{v} \cdot \vec{u})\,\vec{w}$ **(both sides of this identity are vectors)**

Note

For any vectors $\vec{u} = (u_1,u_2,u_3)$, $\vec{v} = (v_1,v_2,v_3)$, $\vec{w} = (w_1,w_2,w_3)$ in $\mathbf{R}^3$:

$$\vec{u}\cdot(\vec{v}\times\vec{w})=\begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}=u_1\begin{vmatrix}v_2 & v_3\\ w_2 & w_3\end{vmatrix}-u_2\begin{vmatrix}v_1 & v_3\\ w_1 & w_3\end{vmatrix}+u_2\begin{vmatrix}v_1 & v_2\\ w_1 & w_2\end{vmatrix}$$

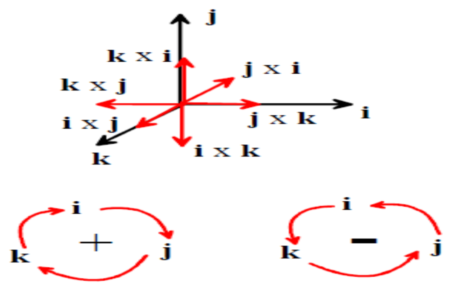
The Cross Product Of i, j And k

$\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$, $\mathbf{k}\times\mathbf{i}=\mathbf{j}$

$\mathbf{i}\times\mathbf{i}=0$, $\mathbf{j}\times\mathbf{j}=0$, $\mathbf{k}\times\mathbf{k}=0$

$\mathbf{i}\times\mathbf{k}=-\mathbf{j}$, $\mathbf{j}\times\mathbf{i}=-\mathbf{k}$, $\mathbf{k}\times\mathbf{j}=-\mathbf{i}$

to find the cross product of any pair of basis vectors ,you travel around the circle. Thus, to get i x j , you start at i, move to j and then on to k. If you go around the circle clockwise, the answer is positive, if you go counter-clockwise, it is negative. Thus, j x k =i , and so on, while k x j =-i , etc.



Example:

Given $\vec{u}$ = i – 2j + k and $\vec{v}$ = 3i + j – 2k, find each of the following.

a. $\vec{u}$ x v **b.** $\vec{v}$ x $\vec{u}$ **c.** $\vec{v}$ x $\vec{v}$

Solution

a. $\vec{u}$ x v $= \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix}$

$= \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} i - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} j + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} k$

$= (4 - 1)\, i - (-2-3) j + (1+6) k = 3i + 5j + 7k$

b. $\vec{v}$ x $\vec{u}$ $= \begin{vmatrix} i & j & k \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} i - \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} j + \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} k$

$= (1 - 4)\, i - (3+2) j + (-6-1) k = -3i - 5j - 7k$

c. $\vec{v}$ x $\vec{v}$ $= \begin{vmatrix} i & j & k \\ 3 & 1 & -2 \\ 3 & 1 & -2 \end{vmatrix}$

$= \begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix} i - \begin{vmatrix} 3 & -2 \\ 3 & -2 \end{vmatrix} j + \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} k$

$= (-2 + 2)\, i - (-6+6) j + (3-3) k = 0i - 0j - 0k = 0$

Example:

Given the vectors $\vec{v} = i - 2j + 4k$ and $\vec{w} = 3i + j - 2k$ find $\vec{v} \times \vec{w}$

Solution

$$\vec{v}\times\vec{w} = \begin{vmatrix} i & j & k \\ 1 & -2 & 4 \\ 3 & 1 & -2 \end{vmatrix}$$

$= i(4 - 4) - j(-2 -12) + k(1+ 6) = 14j + 7k$

$$\vec{w}\times\vec{v} = \begin{vmatrix} i & j & k \\ 3 & 1 & -2 \\ 1 & -2 & 4 \end{vmatrix} = = i(4 - 4) - j(12 + 2) + k(-6-1) = -14j - 7k$$

Example:

Given the vectors $\vec{v} = j + 6k$ and $\vec{w} = i + j$ find $\vec{v}\times\vec{w}$

Solution

$$\vec{v}\times\vec{w} = \begin{vmatrix} i & j & k \\ 0 & 1 & 6 \\ 1 & 1 & 0 \end{vmatrix} = i(1.0 - 6) - j(0 -6) + k(0- 1) = -6i+6j -k$$

Example:

Find $\vec{u}\times(\vec{v}\times\vec{w})$ for $\vec{u} = (1,2,4)$, $\vec{v} = (2,2,0)$, $\vec{w} = (1,3,0)$.

Solution:

Since $\vec{u}\cdot\vec{v} = 6$ and $\vec{u}\cdot\vec{w} = 7$, then

$\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}\cdot\vec{w})\,\vec{v} - (\vec{u}\cdot\vec{v})\,\vec{w}$

$= 7(2,2,0) - 6(1,3,0) = (14,14,0) - (6,18,0)$

$= (8,-4,0)$

Note

Angle between Vectors

Let $\vec{u}$ and $\vec{v}$ be from R^2 or R^3 and let θ be the angle between them. Then

- **The angle between two vectors can be found by using the dot product.**

$$\cos(\theta) = \frac{\vec{u}.\vec{v}}{|u|.|v|} \Rightarrow \theta = \cos^{-1}\left(\frac{\vec{u}.\vec{v}}{|u|.|v|}\right)$$

- **The angle between two vectors can be found by using the cross product.**

$$\sin\theta = \frac{|\vec{v}\times\vec{u}|}{|u||v|} \Rightarrow \theta = \sin^{-1}\left(\frac{|\vec{v}\times\vec{u}|}{|v||u|}\right)$$

- $\vec{u}\cdot\vec{v}$ is $\begin{cases} positive & o \le \theta < \frac{\pi}{2} \\ 0 & \theta = \frac{\pi}{2} \\ negative & \frac{\pi}{2} < \theta \le \pi \end{cases}$

Example

What is the angle in degrees between $\vec{u}$ =(1, 1, 1) and $\vec{v}$ = (2 ,1 , 0),

Solution

$\vec{u}\cdot\vec{v}$ = 1.2 + 1. 1 +1.0= 3

$|u| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{2^2 + 1^2 + 0^2} = \sqrt{5}$

$\cos(\theta) = \frac{\vec{u}.\vec{v}}{|u|.|v|} = \frac{3}{\sqrt{3}\sqrt{5}} = \frac{3}{\sqrt{15}} \Rightarrow \theta = \cos^{-1}\left(\frac{3}{\sqrt{15}}\right)$ =39:23°

Example:

Find the angle between $\vec{u}$. = 2i+3j+k & $\vec{v}$= -i+5j+k.

Solution:

$\vec{u}.\ \vec{v}$=2.(-1)+3.5+1.1 =-2+15+1 =14

$\|\vec{u}\| = \sqrt{14}$, $\|\vec{v}\| = \sqrt{27}$

$\cos(\theta) = \frac{\vec{u}.\vec{v}}{|u|.|v|} = \frac{14}{\sqrt{14}.\sqrt{27}} = \frac{\sqrt{14}}{3\sqrt{3}}$

$\therefore \theta = \cos^{-1}\left(\frac{\sqrt{14}}{3\sqrt{3}}\right)$ =43.93° ≃44°

Example

Given $\vec{u}$ = 2i − 3j + 5k and $\vec{v}$ = 5i + 3j − 7k, compute the angle between $\vec{u}$ and $\vec{v}$

Solution

$\cos\theta = \frac{\vec{u}.\vec{v}}{|u|.|v|}$

$\vec{u} \,.\, \vec{v}$ = 2 x 5 + (-3)x 3 +5x(-7)= -34

$|u| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{2^2 + -3^2 + 5^2} = \sqrt{38}$

$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{5^2 + 3^2 + -7^2} = \sqrt{83}$

$\cos(\theta) = \frac{\vec{u}.\vec{v}}{|u|.|v|} = \frac{-34}{\sqrt{38}\sqrt{83}} = \frac{-34}{\sqrt{3154}}$

Note

One of the application of cross product to find unit vector normal ($\vec{N}$)on $\vec{A}$ and $\vec{B}$

$$\vec{N} = \frac{\vec{A}x\vec{B}}{|\vec{A}x\vec{B}|}$$

Example:

Find the normal unit vector perpendicular on $\vec{A}$ and $\vec{B}$ for

$\vec{A} = 2i + 3j - k$ and $\vec{B} = -j + 2k$

Solution

$$\vec{N} = \frac{\vec{A}x\vec{B}}{|\vec{A}x\vec{B}|}$$

$$\vec{A}\,x\,\vec{B} = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 5i - 4j - 2k$$

$$|\vec{A}\,x\,\vec{B}| = \sqrt{(5)^2 + (-4)^2 + (-2)^2} = \sqrt{45}$$

$$\vec{n} = \frac{5}{\sqrt{45}}i - \frac{4}{\sqrt{45}}j - \frac{2}{\sqrt{45}}k$$

Parallel Vectors

- **Two nonzero vectors $\vec{v}$ and $\vec{u}$ are parallel if there is some scalar c such that**

$$\vec{u} = c\vec{v}$$

Or

- **Two non-zero vectors $\vec{v}$ and $\vec{u}$ are parallel $\Leftrightarrow$ $\vec{v} \times \vec{u} = 0$.**

Example

Which of the following vectors is parallel to $\vec{w}$ = (– 6 , 8, 2)?

a. $\vec{u}$ = (3, –4, –1)

b. $\vec{v}$ =(12, –16, 4)

Solution

a. Because $\vec{u} = (3 ,– 4 , – 1) = \frac{1}{2}(– 6, 8, 2) = -\frac{1}{2}\vec{w}$, you can conclude that u is parallel to w.

OR

$$\vec{w} x \vec{u} = \begin{vmatrix} i & j & k \\ -6 & 8 & 2 \\ 3 & -4 & -1 \end{vmatrix} = (8x\text{-}1 – 2x\text{-}4, 2x3 - \text{-}6x\text{-}1, \text{-}6x\text{-}4\text{-}8x3) = (0 ,\ \ 0,\ \ 0)$$

b. In this case, you want to find a scalar *c* such that

(12, –16, 4) = c(–6, 8, 2) .

12 = –6→ *c* = –2

–16 = 8→ *c* = –2

4 = 2→ *c* = 2

Because there is no *c* for which the equation has a solution ,the vectors are not parallel.

Orthogonal Vector

- **Two non-zero vectors $\vec{v}$ and $\vec{u}$ are orthogonal $\Leftrightarrow$ $\vec{u} \cdot \vec{v} = 0$**

Example

Show that vectors $\vec{v}$=(1, -1, 0) and $\vec{w}$ = (2,2 ,4) are orthogonal,

Solution

$$\vec{v}.\vec{w} = 1*2+-1*2+0*4=0$$

Example

Determine whether the given vectors are orthogonal, parallel or neither:

(1) $\vec{u}$ =(-2, 6, -4) ,$\vec{v}$= (4 ,-12 , 8).

(2) $\vec{u}$=i-j+2k , $\vec{v}$ =2i-j+k.

(3) $\vec{u}$=(a ,b, c) , $\vec{v}$ =(-b, a , 0)

Solution

(1) $\vec{u}$ =(-2 ,6, -4) , $\vec{v}$ = (4, -12, 8).

Because $\vec{v}$ = (4, -12 , 8) = - 2 (-2, 6 ,-4)= 2 $\vec{u}$, you can conclude that $\vec{v}$ is parallel to $\vec{u}$

(2) conclude that $\vec{u}$ is parallel to $\vec{v}$

$\vec{u}.\vec{v}$ = 2+1+2 = 5 ,

$$\vec{u}x\vec{v}=\begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & -1 & 1 \end{vmatrix}= (-1x1 – 2x-1 ,\ \ 1x1 - 2x2\ \ ,\ \ 1x-1- -1x2) =(1 , 3, 1)$$

the vectors are neither orthogonal nor parallel.

(3) $\vec{u}$=(a, b, c) , $\vec{v}$ =(-b ,a , 0).

$\vec{u}$. $\vec{v}$ = -ab + ab + 0 = 0, so the vectors are orthogonal.

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand Theory for vector

c. Central Ideas:

- **Parametric Equations**
- **Vector Equation**

d. Objectives: after the end of courses the student will be able to:

Find

- **Vector Equation**
- **Parametric Equations**

Pre test

Q1- Find the parametric equation of the line passing through $p_0(1,3,2)$ parallel to 2i−j+3k

Parametric Equations

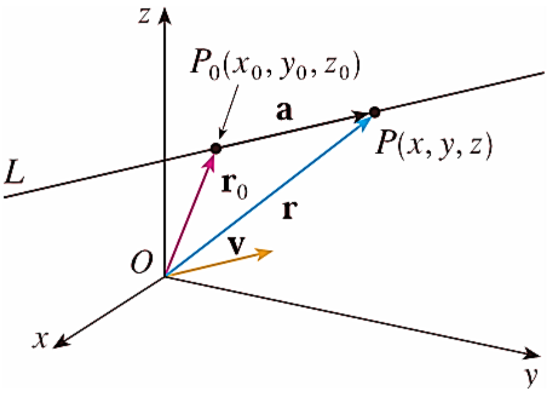
The parametric equations of a lineL in 3-space for a line passing through p_0 (x_0,y_0,z_0) and parallel to $v = ai + bj + ck$:

is $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$

The above equation are called parametric equations for the line

To determine parametric equations of a line, we need

- ✱ a point on the line
- ✱ a vector parallel to the line



Vector Equation

The vector equation of the line is: $\vec{r} = \vec{r_0} + t\,\vec{v}$

Where $\vec{r_0}$ = (x_0, y_0, z_0) is a vector whose components are made of the point (x_0, y_0, z_0) on the line L and

$\vec{v}$ = (a, b, c) are components of a vector that is parallel to the line L

Example:

Find the parametric equation of the line passing through $p_0(1,3,2)$ parallel to $2i-j+3k$.

Solution:

The parametric equation is

$$x = 1 + 2t$$
$$y = 3 - t$$
$$z = 2 + 3t$$

Example:

Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector $\vec{v} = i + 4j - 2k$.

Solution

Here $\overrightarrow{r_0} = (5, 1, 3) = 5i + j + 3k$ and $\vec{v} = i + 4j - 2k$, so the vector

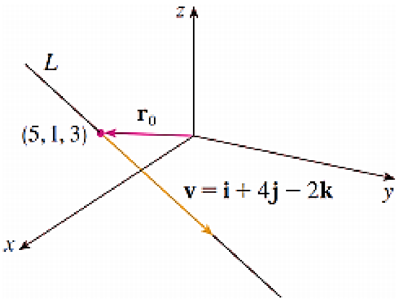
$$\vec{r} = \overrightarrow{r_0} + t\vec{v}$$

becomes

$$\vec{r} = (5i + j + 3k) + t(i + 4j - 2k)$$
$$\vec{r} = (5 + t)i + (1 + 4t)j + (3 - 2t)k$$

Parametric equations are

$$x = 5 + t \quad , \quad y = 1 + 4t \quad , z = 3 - 2t$$



1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand Theory for vector

c. Central Ideas:

- **Symmetric the Equations**
- **Plane of Equation**
- **lines either intersect or are parallel**

d. Objectives: after the end of courses the student will be able to:

1- **Find the relation between parametric equations form equations and symmetricthe Equations**

2- Find the symmetric equation for line

3- find the equation of a plane

4- Show that the lines either intersect or are parallel

Pre test

Q1:Find the equation of the plane with normal $\vec{n}$ = (1, 2, 7) which contains the point P_0 (5, 3, 4)

symmetric the equations

Consider the parametric form equations for a line:

L : $x = x_0 + at$, $y = y_o + bt$, $z = z_0 + ct$.

If a, b and c are all nonzero, we can solve each equation for t to get :

$\frac{x-x_0}{a} = t$

$\frac{y-y_0}{b} = t$

$\frac{z-z_0}{c} = t$

We called these three equation symmetric form of the system of equations for line L, If we set:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = t$$

If one or more of a, b and c is zero, we can still obtain symmetric equations. For example, if a = 0, the symmetric equations are

$$x = x_0 \text{ , } \frac{y-y_0}{b} = \frac{z-z_0}{c} = t$$

Example:

Find the symmetric equation for line through point (1,-5,6) and is parallel to vector (-1,2,-3)

Solution

$$\frac{x-1}{-1} = \frac{y+5}{2} = \frac{z-6}{-3}$$

Equations of Planes

The equation of the plane in 3 space, that passing through the point $P_0(x_0, y_0, z_0)$ on the Plane and the non zero vector $\vec{n} = ai + bj + ck$ Perpendicular (orthogonal) to the plane (The vector $\vec{n}$ is called normal Vector) is $ax + by + cz = D; where\ D = ax_0 + by_0 + cz_0$

Note

- To find the equation of a plane in R^3, we need to know:
  1. A point on the plane $P_0(x_0, y_0, z_0)$.
  2. A normal (perpendicular) vector to the plane.

Example:

Find the equation of plane through point (1,-1,1) and with normal vector i+j-k

Solution

Given point is (1,-1,1)
Here a=1,b=1,c=-1

We know that equation of plane is given by-:

$$ax + by + cz = D; where\ D = ax_0 + by_0 + cz_0$$
$$x + y - z = 1$$

Example: .

Find the equation of the plane with normal $\vec{n}$ = (1, 2, 7) which contains the point P_0 (5, 3, 4)

Solution

The equation of plane is
x+2y+7z=39

Example:

Determine the equation of the plane that contains the points
P_1 = (1, -2 , 0) , P_2 = (3 , 1, 4) , P_3 = (0, -1, 2)

Solution

In order to write the equation of plane we need a point and a normal vector , We need to find a normal vector.

Step 1

First convert the three points into two vectors by subtracting one point from the other two

$\overrightarrow{P_1P_2}$ **= (3-1, 1-(-2) , 4-0) = (2, 3, 4)**
$\overrightarrow{P_1P_3}$ **= (0-1, -1 –(-2) , 2-0) = (-1 , 1, 2)**

Step 2

Find the cross product of the vectors found in Step 1. we know that the cross product of two vectors will be orthogonal to both of these vectors. Since both of these are in the plane any vector that is orthogonal to both of these will also be orthogonal to the plane. Therefore, we can use the cross product as the normal vector.

$$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{vmatrix} = 2i - 8j + 5k$$

Step 3

The coefficients a, b, and c of the planar equation are (2 -8 5), then We can used any of the three points to find The equation of the plane >

2x – 2 -8y -16 +5z =0

2x -8y +5z =18

Example:

Find an equation of the plane which contains the points

P_1 (−1, 0, 1), P_2 (1, -2,1) and P_3 (2, 0, −1).

Solution

Step 1

$\overrightarrow{P_1P_2}$ = (2, -2 , 0)

$\overrightarrow{P_1P_3}$ = (3 , 0 ,-2)

Step 2

$$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} i & j & k \\ 2 & -2 & 0 \\ 3 & 0 & -2 \end{vmatrix} = 4i + 4j + 6k$$

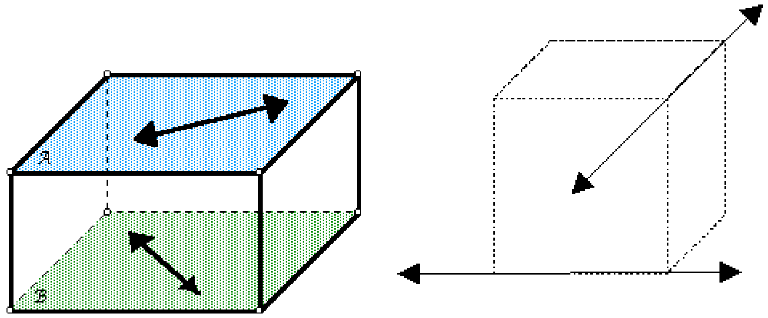
Step 3

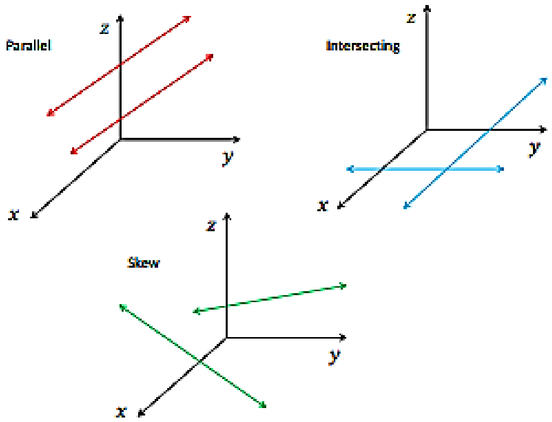
$\vec{n}$ = (4, 4 , 6)

4x + 4 +4y +6z-6 =0

4x +4y +6z =2

In two dimensions, two lines either intersect or are parallel; in three dimensions, lines that do not intersect might not be parallel.
Two lines that are not parallel and do not intersect are called skew lines.





Example:

Show that the lines

L_1: $x = t_1 - 1$, $y = t_1 + 5$, $z = 1$

L_2: $x = t_2 - 3$, $y = -t_2 + 1$, $z = t_2 + 2$

Intersect , and find the point of intersection .

Solution:

If they intersect, we can find a value of t_1 and t_2 that satisfy the equations

$x_0 = t_1 - 1 = t_2 - 3$ -------------------------(1)

$y_0 = t_1 + 5 = -t_2 + 1$----------------------(2)

$z_0 = 1 = t_2 + 2$----------------------------(3)

from equation (3)

$1 = t_2 + 2 \Rightarrow t_2 = -1$

From equation (1) or (2)

$t_1 - 1 = t_2 - 3$

$t_1 - 1 = -1 - 3 \Rightarrow t_1 = -4+1=-3$

Then check whether the three sets of equations are satisfied by (t_2, t_1)

$x_0 = t_1 - 1 = t_2 - 3 \quad \Rightarrow -3-1 = -1-3 \qquad \Rightarrow -4=-4$

$y_0 = t_1 + 5 = -t_2 + 1 \Rightarrow -3+5 = -(-1)+1 \Rightarrow 2 = 2$

$z_0 = 1 = t_2 + 2 \quad \Longrightarrow 1 = -1+2 \quad \Longrightarrow 1 = 1$

The point of intersection $(x_0, y_0, z_0) = (-4, 2, 1)$

Example:

Let L_1 and L_2 be lines with parametric equations

L_1 : $x = 1 + 2t_1$; $y = 3 + 2t_1$; $z = 2 - t_1$

L_2 : $x = 2 + t_2$; $y = 6 - t_2$; $z = -2 + 3t_2$

Determine whether the lines are parallel, skew, or intersecting. If they intersect, find the point of intersection

Solution:

The direction vectors are $\vec{v}_1 = (2; 2; -1)$ and $\vec{v}_2 = (1; -1; 3)$

$\vec{v}_1 \neq c\vec{v}_2$

So these vectors are not parallel . Do they intersect

If there is an intersection point $(x_0; y_0; z_0)$, we will find it by solving the system of three equations in parameters t_1 and t_2:

$x_0 = 1 + 2t_1 = 2 + t_2$ -------------------- (1)

$y_0 = 3 + 2t_1 = 6 - t_2$ -------------------- (2)

$z_0 = 2 - t_1 = -2 + 3t_2$ ------------------- (3)

solve the first two equation for t_1 , t_2 .

$1 + 2t_1 = 2 + t_2$ --------------------- (1)

$3 + 2t_1 = 6 - t_2$ --------------------- (2)

Subtract equation (2) from equation (1) we get

$-2 = -4+2t_2 \Longrightarrow 2t_2 = -2+4$

$t_2 = 1$

We can find t_1 by substituting this value of t_2 in either the first or second equation

$1 + 2t_1 = 2 + t_2 \Longrightarrow 1 + 2t_1 = 2 + 1$

$2t_1 = 2 \Longrightarrow t_1 = 1$

Then check whether the three sets of equations are satisfied by (t_2, t_1)

$x_0 = 1 + 2t_1 = 2 + t_2 \Longrightarrow 1+2 = 2+1 \quad \Longrightarrow 3=3$

$y_0 = 3 + 2t_1 = 6 - t_2 \Longrightarrow 3+2 = 6-1 \Longrightarrow 5 = 5$

$z_0 = 2 - t_1 = -2 + 3t_2 \Longrightarrow 2-1 = -2+3 \quad \Longrightarrow 1 = 1$

The point of intersection $(x_0, y_0, z_0) = (3, 5, 1)$

Example:

Determine whether the lines

L_1 : $x = 1 + 2t_1$; $y = 3t_1$; $z = 2 - t_1$

L_2 : $x = -1 + t_2$; $y = 4 + t_2$; $z = 1 + 3t_2$

parallel, skew or intersecting.

Solution

The direction vectors are $\vec{v}_1 = (2, 3, -1)$ and $\vec{v}_2 = (1, 1, 3)$

$\vec{v}_1 \neq c\vec{v}_2$

So these vectors are not parallel . Do they intersect
If there is an intersection point (x_0; y_0; z_0), we will find it by solving the system of three equations in parameters t_1 and t_2

$x_0 = 1 + 2t_1 \quad = -1 + t_2$ ----------------------- (1)
$y_0 = 3t_1 = 4 + t_2$ ----------------------- (2)
$z_0 = 2 - t_1 = 1 + 3t_2$ ------------------------ (3)

Let us solve the first two equations.

$1 + 2t_1 \; = -1 + t_2$ ----------------------- (1)
$3t_1 \quad = 4 + t_2$ ----------------------- (2)
subtract equation (2) from equation (1) we get
$1 - t_1 = -5 \Longrightarrow \quad t_1 = 6$

we can find t_2 by substituting this value of t_1 in either the first or second equation

$3t_1 \quad = 4 + t_2$
$3(6) \quad = 4 + t_2 \Longrightarrow \; t_2 = 18 - 4 = 14$

Then check whether the three sets of equations are satisfied by (t_2, t_1)
$x_0 = 1 + 2t_1 \quad = -1 + t_2 \Longrightarrow 1 + 2(6) \quad = -1 + 14 \Longrightarrow 13 = 13$
$y_0 = 3t_1 = 4 + t_2 \quad \Longrightarrow \quad 3(6) = 4 + 14 \quad \Longrightarrow \quad 18 = 18$
$z_0 = 2 - t_1 = 1 + 3t_2 \quad \Longrightarrow \quad 2 - 6 = 1 + 3(14) \quad \Longrightarrow -4 \; = 43$

The solution does not satisfy the third equation. So these lines do not intersect, therefore, they are sk

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand Theory for vector

c. Central Ideas:

- **distance *D* from point to the plane**
- **distance *D* from point to the line**
- **distance *D* between two line**

d. Objectives: after the end of courses the student will be able to:

1- **Find the distance *D* from point to the plane**
2- Find the distance *d* from the point *P* to the line
3- Find Distance between two lines

Pre test

Q1: Find the distance from the point p (1,1 ,1) to the line $= \frac{x+3}{7} = \frac{y-1}{3} = \frac{z+4}{-2}$

Q2: Find the distance from the point p(1,1,5) to the line $x - 1 = \frac{y-3}{-1} = \frac{z}{2}$

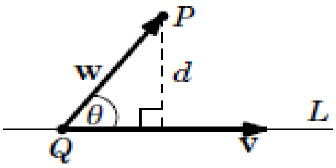
Distance between a Point and a Line

Let L be a line and let P be a point not on L. The distance d from P to L is the length of the line segment from P to L which is perpendicular to L ,Pick a point P_0 on L, and let w be the vector from P_0 to P. If θ is the angle between w and v, then

$d = |w| \sin \theta.$

$\because |\vec{v} \times \vec{w}| = |\vec{v}|\, |\vec{w}| \sin \theta$

$\therefore |\vec{w}| \sin \theta = \frac{|\vec{v} \times \vec{w}|}{|\vec{v}|} \implies d = \frac{|\vec{v} \times \vec{w}|}{|\vec{v}|}$



Example:

Find the distance d from the point $P = (1,1,1)$ to the line

$L : x = -3+7t,\ y = 1+3t,\ z = -4-2t$

Solution

The distance d from P to L :

$d = \frac{|\vec{v} \times \vec{w}|}{|\vec{v}|}$

$\vec{v} = (7, 3, -2)$

$\vec{w} = \overrightarrow{P_0P} = (1-(-3),\ 1-1,\ 1-(-4)) = (4, 0, 5)$

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 7 & 3 & -2 \\ 4 & 0 & 5 \end{vmatrix} = 15i - 43j - 12k$$

$|\vec{v} \times \vec{w}| = \sqrt{15^2 + -43^2 + -12^2} = \sqrt{2218}$

$|\vec{v}| = \sqrt{7^2 + 3^2 + -2^2} = \sqrt{49 + 9 + 4} = \sqrt{62}$

$\therefore d = \frac{|\vec{v} \times \vec{w}|}{|\vec{v}|} = \frac{\sqrt{2218}}{\sqrt{62}} = 5.98$

Example:

Find the distance *d* from the point *P* = (1, 4, -3) to the line

L **: x=2+t, y=-1-t, z=3t**

Solution

The distance *d* from *P* to *L* :

$$d = \frac{|\vec{v} \times \vec{w}\,|}{|\vec{v}|}$$

$\vec{v}$ = (1 ,-1, 3)

$\vec{w} = \overrightarrow{P_0P}$ = (1- 2 , 4- (-1), -3- 0) = (-1 ,5, -3)

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 1 & -1 & 3 \\ -1 & 5 & -3 \end{vmatrix} = -12i - 6j + 6k$$

$$|\vec{v} \times \vec{w}| = \sqrt{(-12)^2 + -6^2 + 6^2} = \sqrt{216}$$

$$|\vec{v}| = \sqrt{1^2 + -1^2 + 3^2} = \sqrt{1+1+9} = \sqrt{11}$$

$$\therefore d = \frac{|\vec{v} \times \vec{w}\,|}{|\vec{v}|} = \frac{\sqrt{216}}{\sqrt{11}} = 4.43$$

Distance between two lines

Let P_1 be a point and $\vec{v_1}$ be a direction vector for a line L_1 and

let P_2 be point and $\vec{v_2}$ be a direction vector for a line L_2 .

The distance between two parallel line

If $\vec{v_1} \times \vec{v_2} = 0$ OR $\vec{v_1} = c\,\vec{v_2} \implies L_1 \parallel L_2$

$$d = \frac{|\vec{v_1} \times \overrightarrow{P_1P_2}\,|}{|\vec{v_1}\,|}$$

The distance between two intersection line

If $\vec{v_1} \times \vec{v_2} \neq 0$ & $\overrightarrow{P_1P_2}.(\vec{v_1} \times \vec{v_2}) = 0 \implies L_1 \cap L_2$

d=0

The distance between two skew line

If $\vec{v_1} \times \vec{v_2} \neq 0$ & $\overrightarrow{P_1P_2}.(\vec{v_1} \times \vec{v_2}) \neq 0 \implies$ the two lines are skew

$$d = \frac{|\overrightarrow{P_1P_2}.(\vec{v_1} \times \vec{v_2})\,|}{|\vec{v_1} \times \vec{v_2}|}$$

Example:

Find the distance *d* between the two lines:

1. **L_1 : x=1+2t, y=2+t, z=-3+3t**

L_2 : x=2+10t, y=-2+5t, z=3+15t

2. L_1 : x=1+2t, y=2+2t, z=-3+3t
L_2 : x=2+t, y=-2-t, z=3+7t
3. L_1 : x=1+t, y=1-2t, z=8+t
L_2 : x=3t, y=2+5t, z=8-8t

Solution

1. L1 : x=1+2t, y=2+t, z=-3+3t
L2 : x=2+10t, y=-2+5t, z=3+15t
The direction vectors are $\vec{v}_1$ = (2 ,1 , 3) and $\vec{v}_2$ = (10, 5 ,15)
$\vec{v}_1 = 5\vec{v}_2$
So these vectors are parallel

$$d = \frac{|\vec{v_1} \times \overrightarrow{P_1P_2}\,|}{|\vec{v_1}|}$$

$\overrightarrow{P_1P_2}$ = (2-1 , -2-2, 3- (-3))= (1 ,-4, 6)

$$\vec{v_1} \times \overrightarrow{P_1P_2} = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 1 & -4 & 6 \end{vmatrix} = 18i - 9j - 9k$$

$$\left|\vec{v_1} \times \overrightarrow{P_1P_2} = \right| = \sqrt{18^2 + -9^2 + -9^2} = 9\sqrt{6}$$

$$|\vec{v_1}| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$$

$$d = \frac{9\sqrt{6}}{\sqrt{14}} = 5.9$$

2. L_1 : x=1+2t, y=2+2t, z=-3+3t
L_2 : x=2+t, y=-2-t, z=3+7t

The direction vectors are $\vec{v}_1$ = (2 , 2, 3) and $\vec{v}_2$ = (1 , -1, 7)
$\vec{v}_1 \neq c\vec{v}_2$
So these vectors are not parallel

$\overrightarrow{P_1P_2}$= (2-1 , -2-2 , 3 – (-3)) = (1 , -4 , 6)

$$\vec{v_1} \text{ x } \vec{v_2} = \begin{vmatrix} i & j & k \\ 2 & 2 & 3 \\ 1 & -1 & 7 \end{vmatrix} = 17i - 11j - 4k$$

$\overrightarrow{P_1P_2}.(\vec{v_1} \text{ x } \vec{v_2})$ = 1 x 17 +(-4 x 11) +6 x (-4) = 17-44-24 =37 $\neq$ 0

$\vec{v_1} \text{ x } \vec{v_2} \neq 0$ & $\overrightarrow{P_1P_2}.(\vec{v_1} \text{ x } \vec{v_2}) \neq 0 \Longrightarrow$ the two lines are skew

$$d = \frac{|\overrightarrow{P_1P_2}.(\vec{v_1} \text{ x } \vec{v_2})\,|}{|\vec{v_1} \text{ x } \vec{v_2}|}$$

$|\overrightarrow{v_1} \times \overrightarrow{v_2}| = \sqrt{17^2 + -11^2 + -4^2} = \sqrt{426}$

$\therefore$ d = $\frac{|37|}{\sqrt{426}} = \frac{37}{\sqrt{426}}$ = 1.79

3. L_1 : x=1+t, y=1-2t, z=8+t

L_2 : x=3t, y=2+5t, z=8-8t

The direction vectors are $\vec{v}_1$ = (1 , -2 , 1) and $\vec{v}_2$ = (3, 5 ,-8)

$\vec{v}_1 \neq c\vec{v}_2$

So these vectors are not parallel

$\overrightarrow{P_1P_2}$= (0-1 ,2-1 , 8–8) = (-1, 1, 0)

$$\overrightarrow{v_1} \times \overrightarrow{v_2} = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 3 & 5 & -8 \end{vmatrix} = 11i + 11j + 11k$$

$\overrightarrow{P_1P_2}.(\overrightarrow{v_1} \times \overrightarrow{v_2})$ = -1 x 11+1 x 11+0 x 11=0

$\overrightarrow{v_1} \times \overrightarrow{v_2} \neq 0$ & $\overrightarrow{P_1P_2}.(\overrightarrow{v_1} \times \overrightarrow{v_2}) = 0 \Rightarrow L_1 \cap L_2$

$\therefore$ **d = 0**

Distance between Point and Plane

The distance D between a point $p_0 = (x_0, y_0, z_0)$ and the plane; $ax + by + cz + d = 0$ is

$$D = \frac{|a\,x_1 + b\,y_1 + c\,Z_1 + d|}{\sqrt{a^2+b^2+c^2}}$$

Example:

Find the distance *D* from point (2,4,−5) to the plane

5*x* - 3*y* + *z*−10= 0

Solution

$$D = \frac{|a\,x_1 + b\,y_1 + c\,Z_1 + d|}{\sqrt{a^2+b^2+c^2}} =$$

$$D = \frac{|5x - 3y + z - 10|}{\sqrt{5^2 + -3^2 + 1^2}} = \frac{|5(2) - 3(4) + (-5) - 10|}{\sqrt{35}} = \frac{|-17|}{\sqrt{35}} = \frac{17}{\sqrt{35}} = 2.87$$

Example:

Find the distance *D* from point(1, 6 , -1)to the plane

2x + y - 2z - 19= 0

Solution

.

$$D = \frac{|a\,x_1 + b\,y_1 + c\,Z_1 + d|}{\sqrt{a^2+b^2+c^2}}$$

$$D = \frac{|2x + y - 2z - 19|}{\sqrt{2^2+1^2+-2^2}} = \frac{|2(1) + 6 - 2(-1) - 19|}{\sqrt{9}} = \frac{|-9|}{\sqrt{9}} = \frac{9}{3} = 3$$

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand Partial Derivatives

c. Central Ideas:

- **Partial Derivatives first and second order**
- **Plane of Equation**
- **lines either intersect or are parallel**

d. Objectives: after the end of courses the student will be able to:

1-Find Partial Derivatives first and second order

2- Find formula for Del operation

Pre test

Q1: Compute grad $\vec{F}$ and div $\vec{F}$ for $\vec{F} = 4x - y^2 e^{3xz}$

Partial Derivatives

a **partial derivative** of a function of several variables is its derivative with respect to one of those variables, with the others held constant (as opposed to the total derivative, in which all variables are allowed to vary). The partial derivative is denoted by:

The partial derivative of f with respect to x. $\frac{\partial f}{\partial x}$

The partial derivative of f with respect to y. $\frac{\partial f}{\partial y}$

The partial derivative of f with respect to z. $\frac{\partial f}{\partial z}$

Example:

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$, if $f(x, y) = x^2 + 3xy + y - z$?

Sol.

$$\frac{\partial f}{\partial x} = 2x + 3y + 0 - 0 = 2x + 3y$$

$$\frac{\partial f}{\partial y} = 0 + 3x + 1 - 0 = 3x + 1$$

$$\frac{\partial f}{\partial z} = 0 + 0 + 0 - 1 = -1$$

Example: Find $\frac{\partial f}{\partial y}$ if $f(x, y) = y \sin xy$?

Sol.

$$\frac{\partial f}{\partial y} = y \frac{\partial(\sin xy)}{\partial y} + \sin xy \frac{\partial(y)}{\partial y}$$

$$= xy \cos xy + \sin xy$$

Second-Order Partial Derivatives

When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by:

$\frac{\partial^2 f}{\partial x^2}$ The second-order partial derivative of f with respect to x.

$\frac{\partial^2 f}{\partial y^2}$ The second-order partial derivative of f with respect to y.

$\frac{\partial^2 f}{\partial z^2}$ The second-order partial derivative of f with respect to z.

Example:

If $f(x, y) = x^2y^3 + xy^2 - z^2$, find $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}$?

Sol.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\left(x^2y^3 + xy^2 - z^2\right)}{\partial x}\right) = \frac{\partial(2xy^3 + y^2)}{\partial x} = 2y^3$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\left(x^2y^3 + xy^2 - z^2\right)}{\partial y}\right) = \frac{\partial(3x^2y^2 + 2xy)}{\partial y} = 6x^2y + 2x$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial\left(x^2y^3 + xy^2 - z^2\right)}{\partial z}\right) = \frac{\partial(-2z)}{\partial z} = -2$$

Formulas for Del Operation

The vector differential operator $\vec{\nabla}$ called "del" , is defined as:

$$\vec{\nabla}=\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}$$

If a scalar function f(*x,y,z*) and vector $\vec{A}=A_1\vec{i}+A_2\,\vec{j}+A_3\,\vec{k}$ have partial derivatives, we can define the following:

1.Gradient Field

The gradient of the function f(*x,y,z*)is define by:

$$grad\ f=\vec{\nabla}f=\frac{\partial f}{\partial x}\vec{i}+\frac{\partial f}{\partial y}\vec{j}+\frac{\partial f}{\partial z}\vec{k}$$ Vector field.

---

2. Divergence Field

The divergence (flux density) of $\vec{A}$ is define by:

$$div\ \vec{A}=\vec{\nabla}\cdot\vec{A}=\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}$$ Scalar field.

3.Curl Field

The curl of the vector $\vec{A}$ is define by:

$$Curl\ \vec{A}=\vec{\nabla}\times\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}$$

$$=\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\vec{i}-\left(\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z}\right)\vec{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\vec{k}$$ Vector field.

4.Laplacian Field

Scalar field. $$\vec{\nabla}\cdot(\vec{\nabla}\,f)=\nabla^2 f=\frac{\partial^2 f}{\partial\,x^2}+\frac{\partial^2 f}{\partial\,y^2}+\frac{\partial^2 f}{\partial\,z^2}$$

<u>Example</u>:

Find the gradient field of the following functions:

a) $f(x,y,z) = xy + x^2 - 2z$ **b)** $\emptyset(x,y,z) = 2x + 3y^2 + \sin z$

<u>Sol.</u>

a) $f(x,y,z) = xy + x^2 - 2z$

$$\vec{\nabla} f(x,y,z) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$= (y + 2x)\vec{i} + (x)\,\vec{j} - 2\vec{k}$$

b) $\emptyset(x,y,z) = 2x + 3y^2 + \sin z$

$$\vec{\nabla}\emptyset(x,y,z) = \frac{\partial \emptyset}{\partial x}\vec{i} + \frac{\partial \emptyset}{\partial y}\vec{j} + \frac{\partial \emptyset}{\partial z}\vec{k}$$

$$= 2\vec{i} + (6y)\,\vec{j} + (\cos z)\vec{k}$$

Example:

Find the divergence field of the following vectors:

a) $\vec{R} = (4xz)\vec{i} + (3x)\,\vec{j} + (5yz)\vec{k}$ **b)** $\vec{A} = (e^x)\vec{i} + (\ln xy)\,\vec{j} + (e^{xyz})\vec{k}$

Sol.

a) $\vec{R} = (4xz)\vec{i} + (3x)\,\vec{j} + (5yz)\vec{k}$

$$\vec{\nabla} \cdot \vec{R} = \frac{\partial R_1}{\partial x} + \frac{\partial R_2}{\partial y} + \frac{\partial R_3}{\partial z}$$

$$= 4z + 0 + 5y \quad = 4z + 5y$$

b) $\vec{A} = (e^x)\vec{i} + (\ln xy)\,\vec{j} + (e^{xyz})\vec{k}$

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$= e^x + \frac{x}{xy} + xy\, e^{xyz} \quad = e^x + \frac{1}{y} + xy\, e^{xyz}$$

Example

Find the Curl field of the following vectors:

a) $\vec{A} = (x^2 - y)\vec{i} + (4z)\,\vec{j} + (x^2)\vec{k}$ **b)** $\vec{B} = (3x^2)\vec{i} + (2z)\,\vec{j} + (\sin x)\vec{k}$

Sol.

a) $\vec{A} = (x^2 - y)\vec{i} + (4z)\,\vec{j} + (x^2)\vec{k}$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x^2 - y) & 4z & x^2 \end{vmatrix}$$

$$= \left(\frac{\partial(x^2)}{\partial y} - \frac{\partial(4z)}{\partial z}\right)\vec{i} - \left(\frac{\partial(x^2)}{\partial x} - \frac{\partial(x^2 - y)}{\partial z}\right)\vec{j} + \left(\frac{\partial(4z)}{\partial x} - \frac{\partial(x^2 - y)}{\partial y}\right)\vec{k}$$

$$= (0 - 4)\vec{i} - (2x - 0)\vec{j} + (0 + 1)\vec{k}$$

$$= -4\,\vec{i} - (2x)\,\vec{j} + \vec{k}$$

b) $\vec{B} = (3x^2)\vec{\imath} + (2z)\,\vec{\jmath} + (\sin x)\vec{k}$

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2 & 2z & \sin x \end{vmatrix}$$

$$= \left(\frac{\partial(\sin x)}{\partial y} - \frac{\partial(2z)}{\partial z}\right)\vec{\imath} - \left(\frac{\partial(\sin x)}{\partial x} - \frac{\partial(3x^2)}{\partial z}\right)\vec{\jmath} + \left(\frac{\partial(2z)}{\partial x} - \frac{\partial(3x^2)}{\partial y}\right)\vec{k}$$

$$= (0-2)\vec{\imath} - (\cos x - 0)\vec{\jmath} + (0-0)\vec{k}$$

$$= -2\,\vec{\imath} - (\cos x)\,\vec{\jmath}$$

Example:

Find the laplacian field of the function $f(x, y, z) = 2x^2y - x\,z^3$?

Sol.

$$\nabla^2 f = \frac{\partial^2 f}{\partial\, x^2} + \frac{\partial^2 f}{\partial\, y^2} + \frac{\partial^2 f}{\partial\, z^2}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial(2x^2y - x\,z^3)}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial(2x^2y - x\,z^3)}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial(2x^2y - x\,z^3)}{\partial z}\right)$$

$$= \frac{\partial(4xy - z^3)}{\partial x} + \frac{\partial(2x^2)}{\partial y} + \frac{\partial(-3xz^2)}{\partial z}$$

$$= (4y) + 0 + (-6xz) \quad = 4y - 6xz$$

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1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand polar coordinate system

c. Central Ideas:

- **polar coordinate system**
- **draw the point in polarcoordinate**
- **convert the polar coordinate in cartisan or from cartisan in polar coordinate**

d. Objectives: after the end of courses the student will be able to:

1- convert the point in polar or in cartisan coordinate
2- draw the point in polar coordinate

Pre test

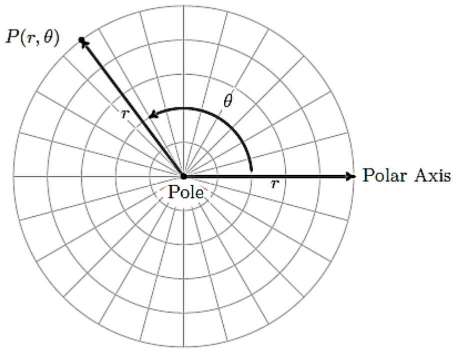
Q1: Find the Cartesian coordinates of the point with the following polar coordinates: $(6, \frac{\pi}{6})$

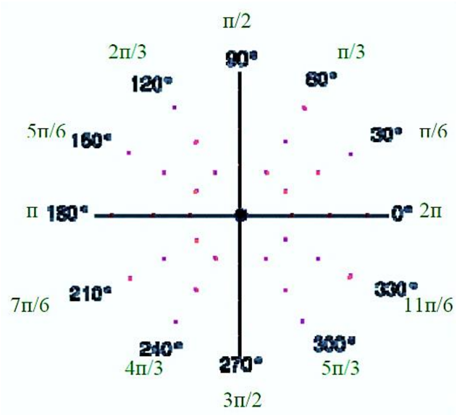
Q2: Find the polar coordinates of the point with the following Cartesian coordinates: (2,2)

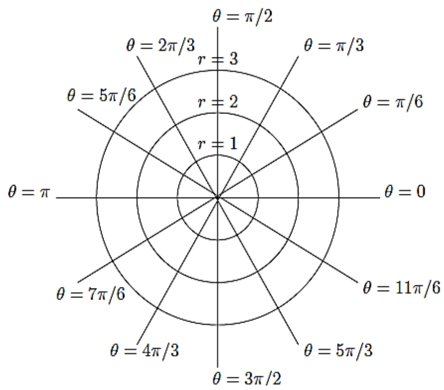
Q3 : draw the point $(4, \frac{\pi}{2})$

Polar Coordinate

The polar coordinate system is a two-dimensional coordinate system in which each point P on a plane is determined by a distance r from a fixed **point O that is called the pole (or origin)** and an angle θ (in degrees or radians)from a fixed direction.
The point P is represented by the ordered pair (r,θ) and (r, θ)are called polar coordinates.
The polar coordinate pair (r, θ) species a point in 2D space as follows:
1. Start at the origin, facing in the direction of the polar axis, and rotate by angle θ. Positive values of θ are usually interpreted to mean counterclockwise rotation ,with negative values indicating clockwise rotation.
2. Now move forward from the origin a distance of r units.



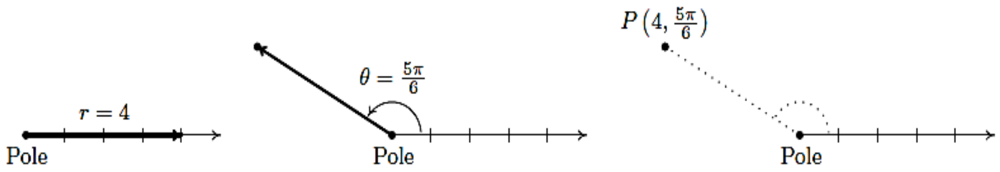




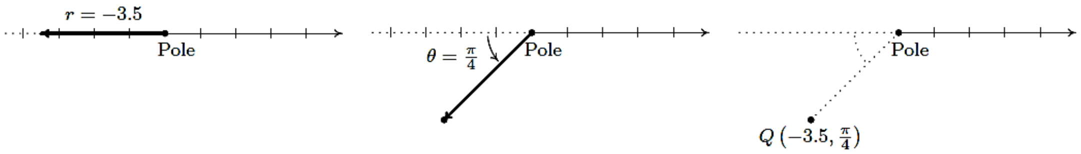
Note

In rectangular coordinate system, each point has unique coordinates but in polar coordinate system a point has infinitely many coordinates.

For example, if we wished to plot the point P with polar coordinates $(4, \frac{5\pi}{6})$ we'd start at the pole ,move out along the polar axis 4 units, then rotate $\frac{5\pi}{6}$ radians counter-clockwise.

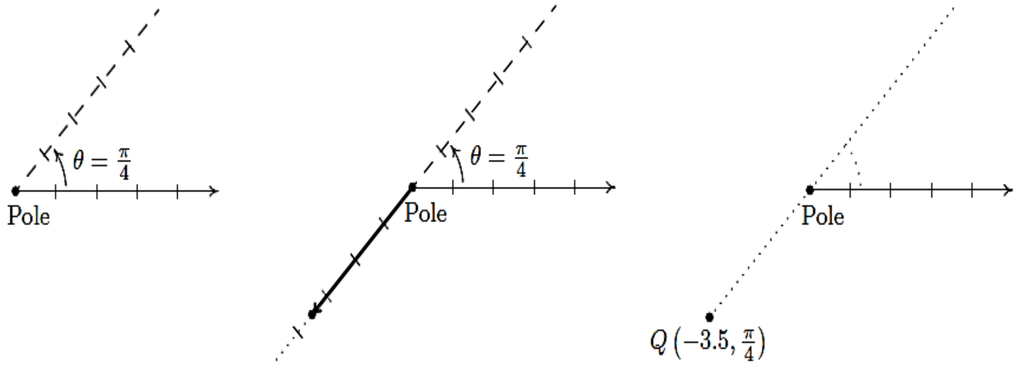


If r < 0, we begin by moving in the opposite direction on the polar axis from the pole. For example,to plot Q(-3.5, $\frac{\pi}{4}$) we have

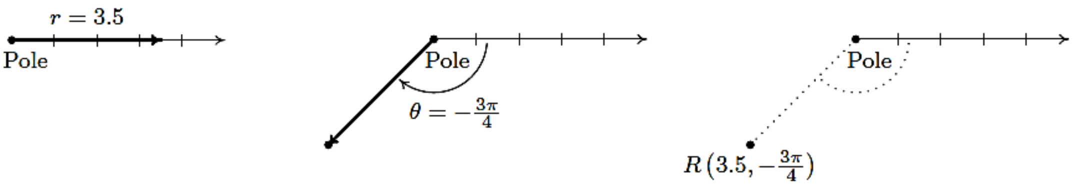


If we interpret the angle first, we rotate $\frac{\pi}{4}$ radians, then move back through the pole 3.5 units.

Here we are locating a point 3.5 units away from the pole on the terminal side of $\frac{5\pi}{4}$, not $\frac{\pi}{4}$.

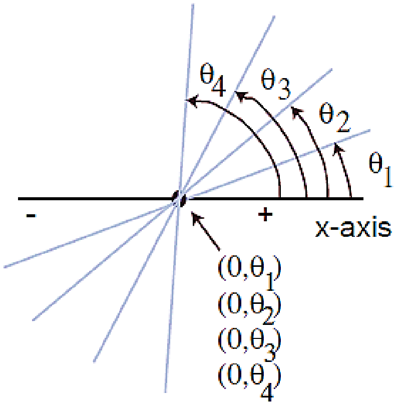


As you may have guessed, $\theta < 0$ means the rotation away from the polar axis is clockwise instead of counter-clockwise. Hence, to plot R (3.5,$\frac{-3\pi}{4}$) we have the following.

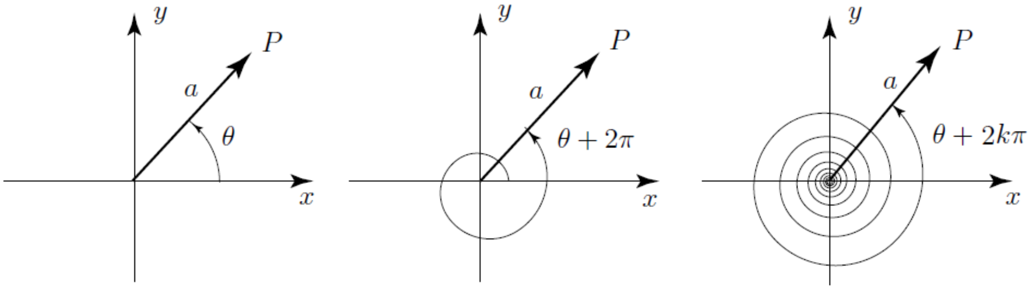


Points in Polar Coordinates

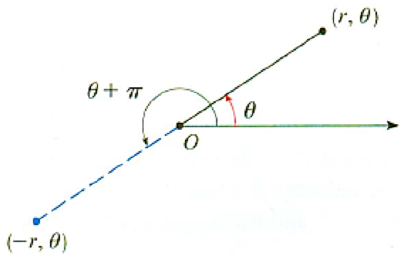
- O = (0,θ) for allθ (This is so because for any θ the point that is distance 0 away from the origin along the line L is the origin).



- (r, θ) = (r, θ + $2k\pi$) for all integers k.



- (-r, θ) = (r, θ +π).

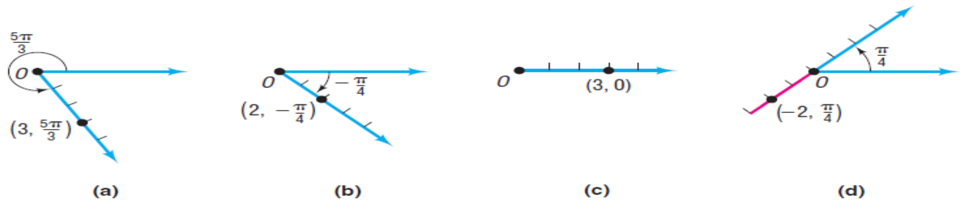


Example:

Plotting Points Using Polar Coordinates

a. $(3 , \frac{5\pi}{3})$
b. $(2 , -\frac{\pi}{4})$
c. (3,0)
d. $(-2 , \frac{\pi}{4})$

Solution



Converting from Polar Coordinates to Cartesian Coordinates

If P is a point with polar coordinates (r , θ) the Cartesian coordinates (x , y) of P is given by:

$$x = r\cos\theta \qquad y = r\sin\theta$$

Note

If r =0 , then , regardless of θ , the a point P with Cartesian coordinates are (0,0)

Example

Find the Cartesian coordinates of the points with the following polar coordinates:

(a) (6 , $\frac{\pi}{6}$) **(b) (-4 , -$\frac{\pi}{4}$)**

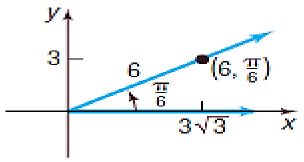
Solution:

(a)

$x = r\cos\theta = 6\cos(\frac{\pi}{6}) = 6\frac{\sqrt{3}}{2} = 3\sqrt{3}$

$y = r\sin\theta = 6\sin(\frac{\pi}{6}) = 6\frac{1}{2} = 3$

The rectangular (Cartesian) coordinates of the point (6, $\frac{\pi}{6}$) are ($3\sqrt{3}$, 3)

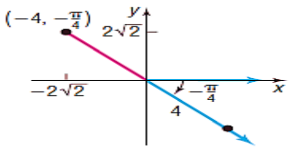


(b)

$x = r\cos\theta = -4\cos(-\frac{\pi}{4}) = -4\frac{1}{\sqrt{2}} = -2\sqrt{2}$

$y = r\sin\theta = -4\sin(-\frac{\pi}{4}) = -4(-\frac{1}{\sqrt{2}}) = 2\sqrt{2}$

The rectangular (Cartesian) coordinates of the point (-4, -$\frac{\pi}{4}$) are ($-2\sqrt{2}$, $2\sqrt{2}$)



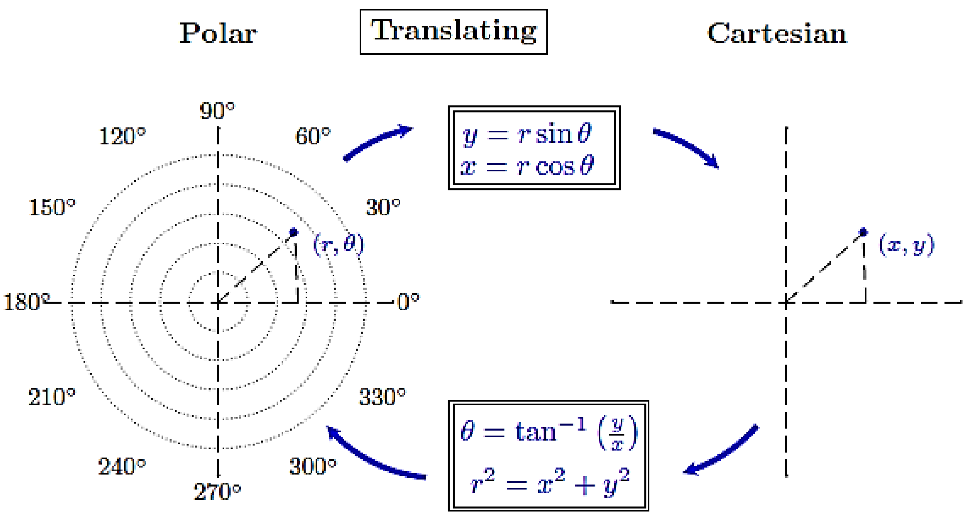
Convert from Cartesian Coordinates to Polar Coordinates

If P is a point with Cartesian coordinates (x , y) the polar coordinates (r , θ) of P is given by:

$$r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$



Steps for Converting from Rectangular to Polar Coordinates

STEP 1: Always plot the point (x, y) first

STEP 2: If $x = 0$ or $y = 0$, use your illustration to find (r, θ).

STEP 3: If $x \neq 0$ and $y \neq 0$, then $r = \sqrt{x^2 + y^2}$.

STEP 4: To find θ, first determine the quadrant that the point lies in.

Quadrant I: $\theta = \tan^{-1}\frac{y}{x}$ Quadrant II: $\theta = \pi + \tan^{-1}\frac{y}{x}$

Quadrant III: $\theta = \pi + \tan^{-1}\frac{y}{x}$ Quadrant IV: $\theta = \tan^{-1}\frac{y}{x}$

Example

Find the polar coordinates of the points with the following Cartesian coordinates:

a) (2,2) b) (-1,1) c) (1,−1) d)(-2,-2$\sqrt{3}$)

Solution:

a) (x ,y) =(2,2)

$r = \sqrt{x^2 + y^2}$

$r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$\tan(\theta) = \frac{y}{x}$

$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$

$\because$ x > 0 , y > 0 $\Longrightarrow$ the first quadrant

$\therefore (r\ ,\theta) = (2\sqrt{2}\ ,\frac{\pi}{4})$

b) (x ,y) =(-1,1)

$r = \sqrt{x^2 + y^2}$

$r = \sqrt{-1^2 + 1^2} = \sqrt{2}$

$\tan(\theta) = \frac{y}{x}$

$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = -\frac{\pi}{4}$

$\because$ x < 0 , y > 0 $\Longrightarrow$ the second quadrant, $\frac{\pi}{2} < \theta \leq \pi$

$\Longrightarrow \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\therefore (r\ ,\theta) = (\sqrt{2}, \frac{3\pi}{4})$

c) (x ,y) =(1,-1)

$r = \sqrt{x^2 + y^2}$

$r = \sqrt{1^2 + -1^2} = \sqrt{2}$

$\tan(\theta) = \frac{y}{x}$

$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = -\frac{\pi}{4}$

$\because$ x > 0 $\Longrightarrow$ the fourth quadrant , $\theta = -\frac{\pi}{4}$

$\therefore (r\ ,\theta) = (\sqrt{2}, -\frac{\pi}{4}) = (\sqrt{2}, \frac{7\pi}{4})$

d) (x ,y) = (-2,-2$\sqrt{3}$)

$r = \sqrt{x^2 + y^2}$

$r = \sqrt{-2^2 + (-2\sqrt{3})^2} = \sqrt{4 + 12} = 4$

$\tan(\theta) = \frac{-2\sqrt{3}}{-2}$

$\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$\because$ $x < 0$, $y < 0 \Longrightarrow$ the third quadrant $\theta = (\pi + \frac{\pi}{3}) = \frac{4\pi}{3}$

$\therefore (r , \theta) = (4, \frac{4\pi}{3})$

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand Partial Derivatives

c. Central Ideas:

- **Test Polar Equations for Symmetry**
- **Graph of a polar equation**
- **Steps for Sket ching Polar Equations**

d. Objectives: after the end of courses the student will be able to:

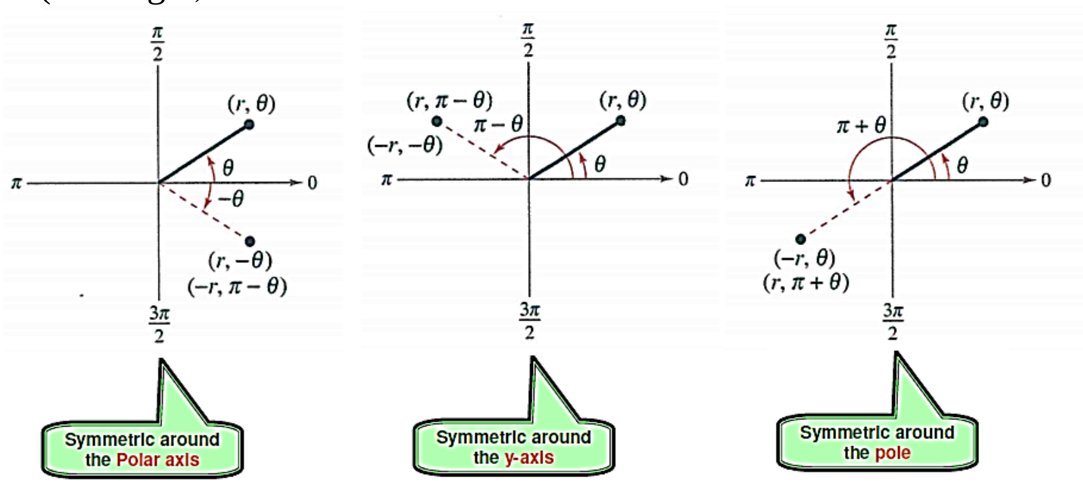
Graph of a polar equation

Pre test
Q1: Graph the polar equations: r = 3cos2θ

Test Polar Equations for Symmetry

In polar coordinates, the symmetry tests for polar graphs:

1. Symmetry about the x-axis: the points(r,θ) and $(r,-\theta)$ are symmetric with respect to the polar axis (the *x*-axis).
2. Symmetry about the y-axis: The points (r,θ) and $(-r,-\theta)$ are symmetric with respect to the line $\theta = \frac{\pi}{2}$ (the *y*-axis).
3. Symmetry about the origin: The points (r,θ) and $(-r,\theta)$ are symmetric with respect to the pole (the origin).



Graph of a polar equation

Polar equation is an equation whose variables are r and θ.
The **graph of a polar** equation is the set of all points whose polar coordinates satisfy the equation.

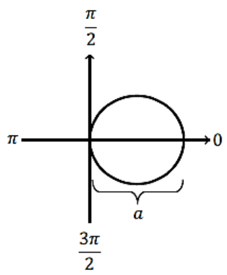
Special Polar Graphs

1- Circles

The graphs of

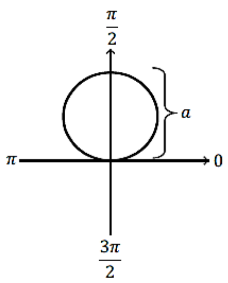
$$r = a\cos\theta, \quad r = a\sin\theta, \quad a \neq 0.$$

are called circles .



$r = a\cos(\theta)$

Circle



$r = a\sin(\theta)$

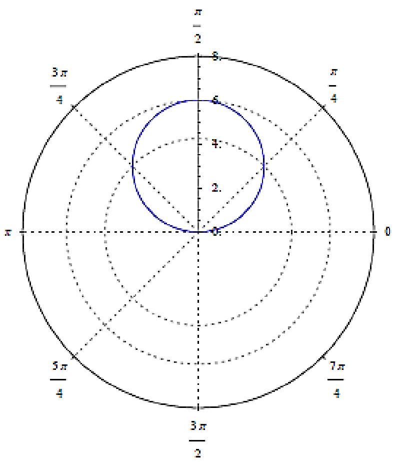
Circle

Example

Sketch the graph of the polar equation $r = 6\sin\theta$.

Solution:

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
$r = 6\sin\theta$	0	4	6	4

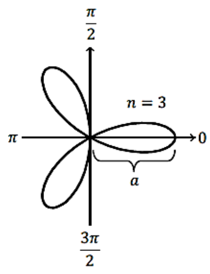


2-Rose curves:

The graphs of

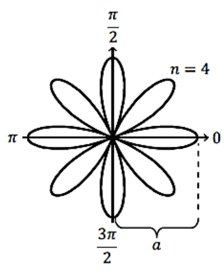
$$r = a\cos n\theta, \quad r = a\sin n\theta, \quad a \neq 0.$$

are called rose curves. If n is even, the rose has 2n petals. If n is odd, the rose has n petals.



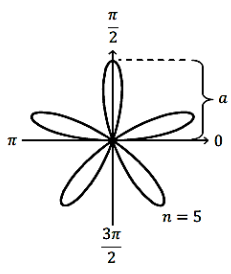
$r = a\cos(n\theta)$

Rose curve



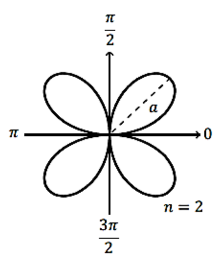
$r = a\cos(n\theta)$

Rose curve



$r = a\sin(n\theta)$

Rose curve



$r = a\sin(n\theta)$

Rose curve

Steps for Sketching Polar Equations- Roses

Step 1

Identify the number of "petals".

•If n is even, then there are $2n$ petals.

•If n is odd, then there are n petals.

Step 2

Determine the length of each petal.

•The length of each petal is $|a|$ units.

Step 3

Determine all angles where an endpoint of a petal lies.

•If the equation is of the form r = a sin nθ, then the endpoints occur for angles on the interval $[0, 2\pi)$

•If the equation is of the form r= a cos nθ, then the endpoints occur for angles on the interval $[0, 2\pi)$

- Note that when n is odd, it is only necessary to consider angles on the interval $[0, \pi)$.

  A complete graph is obtained on this interval because the graph will completely traverse itself on the interval $[\pi, 2\pi)$.

Step 4

Substitute each angle determined in Step 3 back into the original equation to obtain the appropriate values of r for each angle. The ordered pairs obtained represent the endpoints of the rose petals.

Plot these points on the graph.

Step 5

Determine angles where the graph passes through the pole. These angles serve as a guide when sketching the width of a petal.

•If the equation is of the form $r = a\sin n\theta$, then the graph passes through the pole when $\sin n\theta = 0$.

•If the equation is of the form $r = a\cos n\theta$, then the graph passes through the pole when $\cos n\theta = 0$.

Step 6

Draw each petal to complete the graph

Example

Graph the polar equations: r = 2cos3θ.

Solution:

Step 1

n =3 then then there are 3 petals.

Step 2

The length of each petal is2 units.

Step 3

r = 2cos3θ.

2= 2cos3θ.

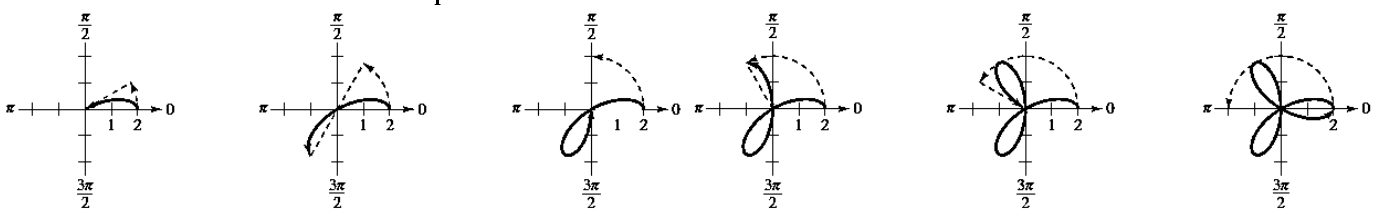
1= cos3θ

$\cos^{-1}(1) = 3\theta$

$0 = 3\theta \Longrightarrow \theta = 0$

$\therefore 2 = 2\cos 3(0)$.

Rotate petals : $\frac{(2\pi)360}{\text{number of petals}} = \frac{360}{3} = \frac{2\pi}{3}$



$0 \le \theta \le \frac{\pi}{6}$ $0 \le \theta \le \frac{\pi}{3}$ $0 \le \theta \le \frac{\pi}{2}$ $0 \le \theta \le \frac{2\pi}{3}$ $0 \le \theta \le \frac{5\pi}{6}$ $0 \le \theta \le \pi$

Example

Graph the polar equations: r = 3cos2θ.

Solution:

(r, - θ) ⟹ r = 3cos2 (-θ) ⟹ r = 3cos2 (θ)

x-axis symmetry: yes

(-r, - θ) ⟹ - r = 3cos2 (-θ) ⟹ -r= 3cos2 (θ) ⟹ r = -3cos2 (θ)

y-axis symmetry: no

(-r, θ) ⟹ - r = 3cos2 (θ) ⟹ - r = 3cos2 (θ) ⟹ r = -3cos2 (θ)

symmetry with respect to the origin :no

Step 1

n =2 then then there are(2 * n= 2 * 2 =4) 4 petals.

Step 2

The length of each petal is 3 units.

Step 3

$r = 3\cos 2\theta$

$3 = 3\cos 2\theta$

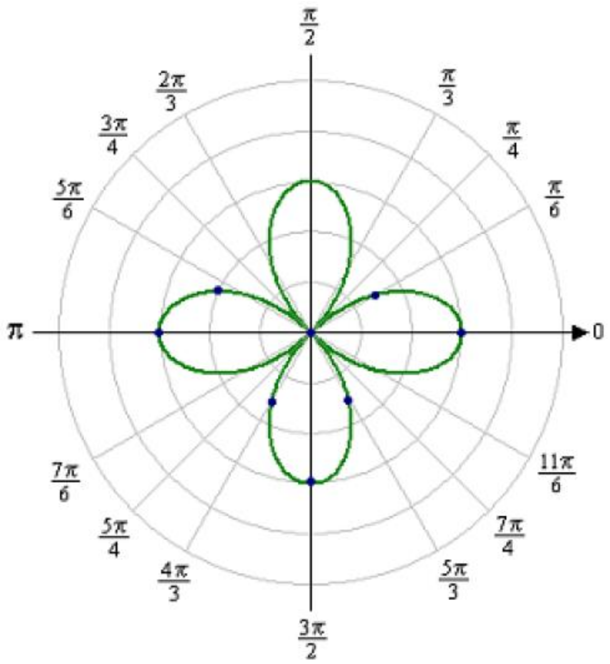
$1 = \cos 2\theta$

$\cos^{-1}(1) = 2\theta$

$0 = 2\theta \Longrightarrow \theta = 0$

$\therefore 3 = 3\cos 2(0)$.

Rotate petals : $\frac{(2\pi)360}{\text{number of petals}} = \frac{360}{4} = \frac{\pi}{2}$

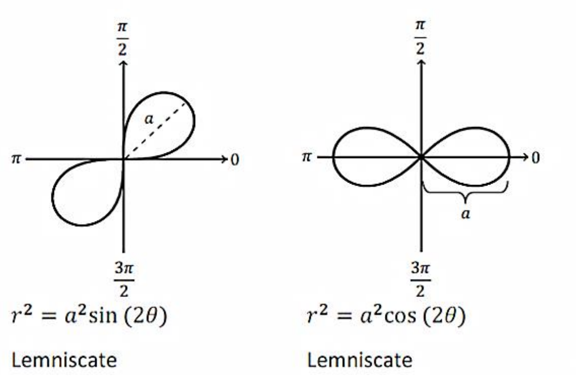


3-Lemniscates

The graphs of

$$r^2 = a^2 \sin 2\theta, \quad r^2 = a^2 \cos 2\theta , \quad a \neq 0.$$

are called Lemniscates.



Example

Graph the polar equations: $r^2 = 4\sin 2\theta$.

Solution:

$(r, -\theta) \Longrightarrow r^2 = 4\sin 2(-\theta) \Longrightarrow r^2 = -4\sin 2(\theta)$

x-axis symmetry: no

$(-r, -\theta) \Longrightarrow (-r)^2 = 4\sin 2(-\theta) \Longrightarrow r^2 = -4\sin 2(\theta)$

y-axis symmetry: no

$(-r, \theta) \Longrightarrow (-r)^2 = 4\sin 2(-\theta) \Longrightarrow r^2 = 4\sin 2(\theta)$

symmetry with respect to the origin :yes

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$r = 2\sqrt{\sin 2\theta}$	0	1.86	2	1.86	0

4-Limacons (Snails)

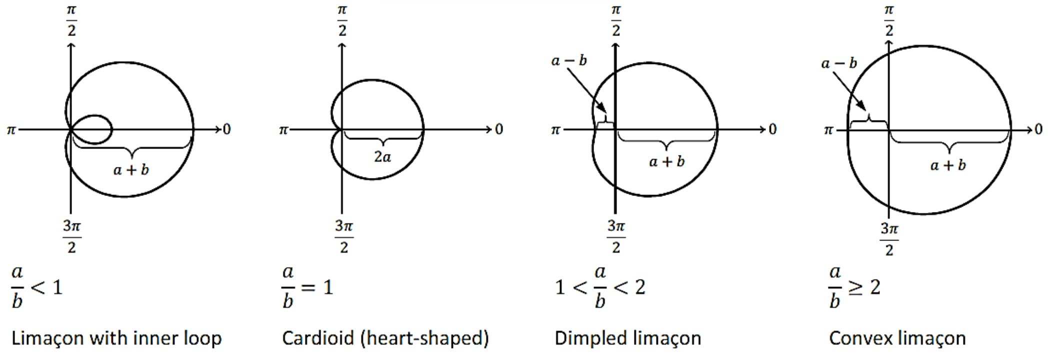
The graphs of

$$r = a + b\cos\theta, \quad r = a - b\cos\theta$$

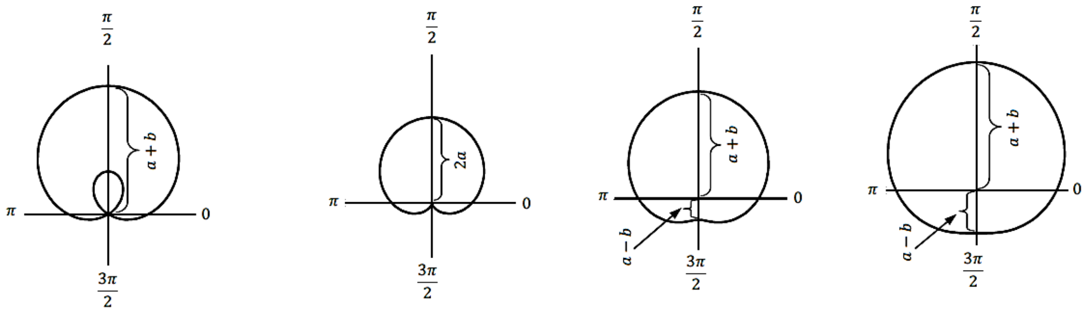
$$r = a + b\sin\theta, \quad r = a - b\sin\theta, \; a > 0, \; b > 0$$

Are called Limacons the ratio $\frac{a}{b}$ determines a Limacons shape

$r = a \pm b\cos\theta$



$r = a \pm b \sin\theta$



$\frac{a}{b} < 1$ Limaçon with inner loop

$\frac{a}{b} = 1$ Cardioid (heart-shaped)

$1 < \frac{a}{b} < 2$ Dimpled limaçon

$\frac{a}{b} \geq 2$ Convex limaçon

Steps for Sketching Polar Equations (Limacons)

Step 1

Identify the general shape using the ratio $\left|\frac{a}{b}\right|$.

•If $\left|\frac{a}{b}\right| = 1$, then the graph is **a cardioid.**

•If $\left|\frac{a}{b}\right| < 1$, then the graph is **a limacon with an inner loop** that intersects the pole.

•If $1 < \left|\frac{a}{b}\right| < 2$, then the graph is **a limacon with a dimple** .

.•If $\left|\frac{a}{b}\right| \geq 2$, then the graph is **a limacon with no inner loop and no dimple**.

Step 2

Determine the symmetry.

•If the equation is of the form $r = a + b\sin\theta$, then the graph must be symmetric about the line $\theta = \frac{\pi}{2}$.

•If the equation is of the form $r = a + b\cos\theta$, then the graph must be symmetric about the polar axis.

Step 3

Plot the points corresponding to the quadrant angles $\theta = 0$, $\theta = \frac{\pi}{2}$, $\theta = \pi$, and $\theta = \frac{3\pi}{2}$.

Step 4

If necessary, plot a few more points until symmetry can be used to complete the graph.

Example

Identify the symmetries of the curve r = 2 + 2 cos θ and then sketch the graph.

Solution:

Step 1

the ratio $\left|\frac{a}{b}\right| = 1 \Longrightarrow$ the graph is a cardioid

Step 2

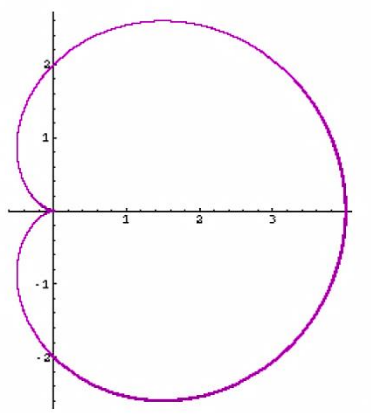
then the graph must be symmetric about the polar axis

Step 3

a+b = 2+2 =4 ⟹stretches on x-axis

a=2 ⟹ stretches on y- axis

θ	*0*	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2 π
r = 2 + 2 cos θ	4	2	0	2	4



Example

Identify the symmetries of the curve r = 1 - 2 cos θ and then sketch the graph.

Solution:

(r, - θ) ⟹ r = 1 - 2 cos (-θ) ⟹ r = 1 - 2 cos θ

x-axis symmetry: yes

(-r, - θ) ⟹ - r = 1 - 2 cos (-θ) ⟹ -r= 1 - 2 cos (θ) ⟹ r = -1 + 2 cos (θ) y-axis symmetry: no

(-r, θ) ⟹ - r = 1 - 2 cos θ ⟹ r = -1 + 2 cos (θ)

symmetry with respect to the origin :no

Step 1

the ratio $\left|\frac{a}{b}\right|$ = 0.5 ⟹ then the graph is a limacon with an inner loop that intersects the pole.

Step 2

then the graph must be symmetric about the polar axis

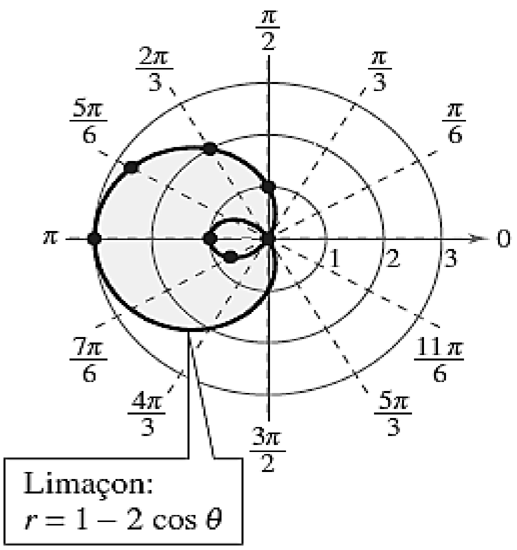
Step 3

a+b = 1+2 =3 ⟹stretches on x- axis

a=2 ⟹ stretches on y- axis

a-b=1-2=-1 ⟹ lower point

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r = 1 - 2 cosθ	-1	0	1	2	$1+\sqrt{3}$	3



1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand polar coordinate

c. Central Ideas:

Areas in Polar Coordinates

d. Objectives: after the end of courses the student will be able to:

1-Find Areas in Polar Coordinates

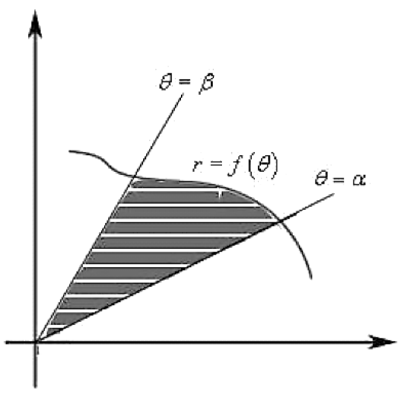
Pre test

Q1: Find the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside the circle $r = 2$

Areas in Polar Coordinates

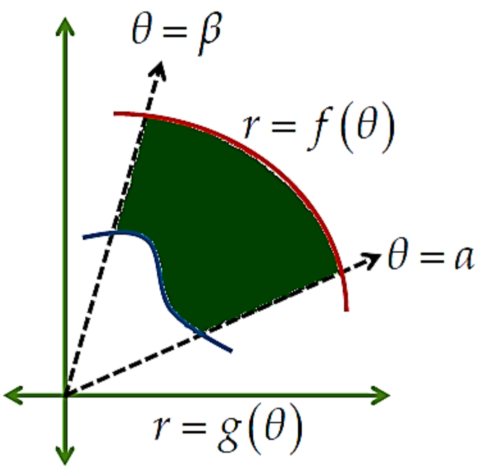
1. If f(θ) be nonnegative continuous function on [α , β] ,then the are A anclosed by polar curve r = f(θ) and lines (rays) $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\, d\theta$$



2. Let R be the region enclosed by nonnegative continuous functions f (θ) and g(θ) on [α , β] , and the lines $\theta = \alpha$ and $\theta = \beta$ is , then the area A of the region R is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} \left|f^2(\theta) - g^2(\theta)\right| d\theta$$

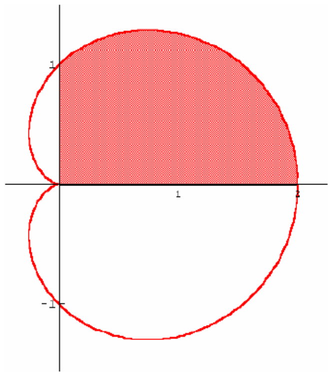


Example:

Find the area in the first quadrant that lies within the curve r = 1 + cos $\boldsymbol{\theta}$.

Solution

the graph of polar equation is cardioid



the area that lies between the rays $\theta = 0 , \frac{\pi}{2}$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + \cos\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + 2\cos\theta + cos^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + 2\cos\theta + [\frac{1+\cos 2\theta}{2}])\, d\theta$$

$$= \frac{1}{2}(\theta + 2\sin\theta + \frac{1}{2}(\theta + \frac{1}{2}\sin 2\theta\Big|_0^{\frac{\pi}{2}}$$

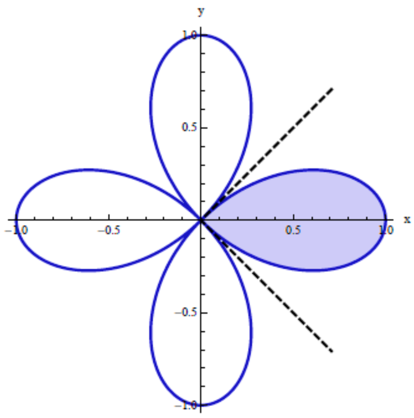
$$= \frac{1}{2}[\frac{\pi}{2} + 2\sin\frac{\pi}{2} + \frac{1}{2}(\frac{\pi}{2} + \frac{1}{2}\sin 2\frac{\pi}{2}) - (0 + 2\sin 0 + \frac{1}{2}(0 + \frac{1}{2}\sin 0))]$$

$$= \frac{1}{2}[\frac{\pi}{2} + 2 + \frac{\pi}{4} + 0) - (0 + 0 + 0))] = \frac{\pi}{4} + 1 + \frac{\pi}{8} = 3\frac{\pi}{8} + 1$$

Example:

Find the area enclosed by one leaf of the 4-leafed rose r = cos (2θ).

Solution



The leaf pointing east is formed by the curve r = cos(2 θ) between two angles for which
r = 0.

$0 = \cos(2\ \theta)$

$\text{Cos}^{-1}(0) = \frac{\pi}{2}$

$2\ \theta = \frac{\pi}{2} \implies \theta = \frac{\pi}{4}$

$\therefore \theta = \pm\ \frac{\pi}{4}$

$A = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2\ d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos 2\theta)^2\ d\theta$

$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta\ d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{4}(1 + cos4\theta)\ d\theta$

$= \frac{\theta}{4} + \frac{1}{16}\sin 4\ \theta \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{16} - (-\frac{\pi}{16}) + \frac{1}{16}\sin \pi - \frac{1}{16}\sin(-\pi) = \frac{\pi}{8}$

Example:

Consider the polar curves r = 6 sinθ and r = 2 + 2 sinθ $0 \le \theta \le 2\pi$.

(a) Find all points of intersection of the two curves.

(b) Graph the two curves and indicate their points of intersection.

(c) Find the area inside the first curve and outside the second.

Solution

(a) Begin by solving the equations simultaneously.

$6 \sin\theta = 2 + 2 \sin\theta$

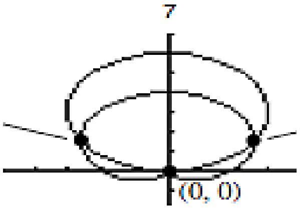
$6 \sin\theta - 2 \sin\theta = 2$

$4 \sin\theta = 2 \implies \sin\theta = \frac{1}{2}$

$\theta = \sin^{-1}(\frac{1}{2}) \implies \theta = \frac{\pi}{6}$

the two polar graphs intersect at (r, θ) = $(3, \frac{\pi}{6})$, $(3,5\ \frac{\pi}{6})$, (0,0)

(b)



(c) The area is given by the following integral.

$$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{5\frac{\pi}{6}} r^2\, d\theta = \int_{\frac{\pi}{6}}^{5\frac{\pi}{6}} [(6\sin\theta)^2 - (2 + 2\sin\theta)^2]\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{5\frac{\pi}{6}} [36\sin^2\theta - (4 + 8\sin\theta + 4\sin^2\theta)]\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{5\frac{\pi}{6}} [36\sin^2\theta - 4 - 8\sin\theta - 4\sin^2\theta]\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{5\frac{\pi}{6}} [16\sin^2\theta - 2 - 4\sin\theta]\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{5\frac{\pi}{6}} [16\frac{(1-\cos 2\theta)}{2} - 2 - 4\sin\theta]\, d\theta = \int_{\frac{\pi}{6}}^{5\frac{\pi}{6}} [8 - 8\cos 2\theta - 2 - 4\sin\theta]\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{5\frac{\pi}{6}} [6 - 8\cos 2\theta - 4\sin\theta]\, d\theta = 6\theta - 4\sin 2\theta + 4\cos\theta \Big|_{\frac{\pi}{6}}^{5\frac{\pi}{6}}$$

$$= 6\left(5\frac{\pi}{6}\right) - 4\sin 2\left(5\frac{\pi}{6}\right) + 4\cos 5\frac{\pi}{6} - \left[6\left(\frac{\pi}{6}\right) - 4\sin 2\left(\frac{\pi}{6}\right) + 4\cos\frac{\pi}{6}\right] = 4\pi$$

Example:

Find the area Outside r = 1+cos θ in side r = $\sqrt{3}$ sin θ.

Solution

$1 + \cos\theta = \sqrt{3}\sin\theta$

$1 + 2\cos\theta + \cos^2\theta = 3\sin^2\theta$

$1 + 2\cos\theta + \cos^2\theta = 3(1 - \cos^2\theta)$

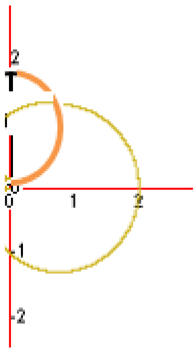
$4\cos^2\theta + 2\cos\theta - 2 = 0$

$2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2} \Rightarrow \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$

$\cos\theta = -1 \Rightarrow \cos^{-1}(-1) = \pi \Rightarrow \theta = \pi$



$A = \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} r^2\, d\theta$

$$=\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}[(\sqrt{3}\ \sin\theta)^2-(1+\cos)^2]d\theta$$

$$=\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}(3\sin^2\theta-1-2\cos\theta-\cos^2\theta)d\theta$$

$$=\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}[\frac{3}{2}(1-\cos2\theta)-1-2\cos\theta-\frac{1}{2}(1+\cos2\theta)]d\theta$$

$$=\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}-2\cos\theta-2\cos2\theta)]d\theta$$

$$=\frac{1}{2}[-2\sin\theta-\sin2\theta\ \Big|_{\frac{\pi}{3}}^{\pi}=\frac{1}{2}[2\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}]=\frac{3\sqrt{3}}{2}$$

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand type of coordinate

c. Central Ideas:

- **cylindricasl coordinate**
- **spherical coordinate**

d. Objectives: after the end of course sylinrical coordinate the student will be able to:

convert from cartisain to cylindricasl coordinate or to cylindricasl coordinate

Pre test

Q1: Given $p(r = 6, \theta = 120, z = -3)$ and $q(x = 5, y = -\sqrt{3}, z = 4)$

Find the length and a unit vector along $\vec{A}$ directed from a point p and q .

Q2:Convert the points from rectangular to spherical coordinates.

a) $(1,-1,-\sqrt{2})$

Cylindrical Coordinate System

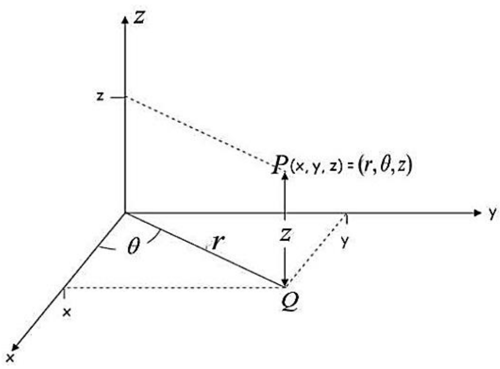
The cylindrical coordinate system basically is a combination of the polar coordinate system xy - plane with an additional z - coordinates vertically.

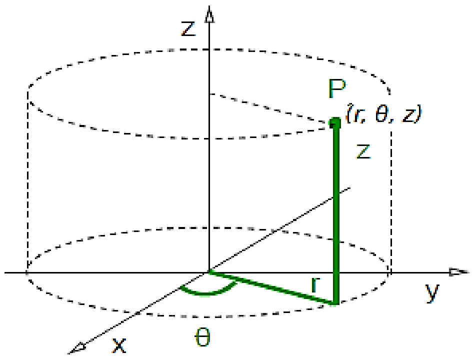
In the cylindrical coordinate system, a point P (x, y, z) ; whose Cartesian Coordinate is (x, y, z) ; is assigned by the ordered triple (r , θ, z) ,

where (r , θ) is the polar coordinate of (x, y) ; the vertical projection along z - axis of P onto xy – plane.

Thus, the **transformation** from **the Cartesian coordinates** to the **cylindrical coordinates** is given by

x = r cos θ
y = r sin θ
z = z





Where $0 \leq r < \infty$; 0 ; $0\leq \theta \leq 2\pi$; $-\infty < z < \infty$

Example:

Convert from cylindrical coordinates (2, $2\frac{\pi}{3}$, 1) to rectangular coordinates.

Solution

To find its rectangular coordinates, we use the formula

$x = r\cos\theta \Longrightarrow x = 2\cos(2\frac{\pi}{3}) = 2\,(\frac{-1}{2}) = -1$

$y = r\sin\theta \Longrightarrow y = 2\sin(2\frac{\pi}{3}) = 2\,(\frac{\sqrt{3}}{2}) = \sqrt{3}$

The point is$(-1\ ,\sqrt{3}\ ,1)$

Note

To convert from rectangular coordinates to cylindrical coordinates.

$r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}(\frac{y}{x})$)

z= z

Example:

Convert the point (-1 , 1 , $\sqrt{2}$) from Cartesian to cylindrical coordinates.

Solution

$r = \sqrt{x^2 + y^2} \Longrightarrow r = \sqrt{-1^2 + 1^2} = \sqrt{2}$

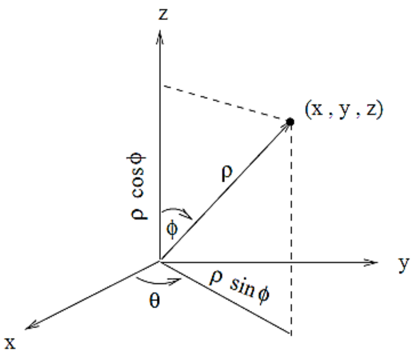
$\tan\theta = \frac{y}{x} \Longrightarrow \theta = \tan^{-1}(\frac{1}{-1}) = -\frac{\pi}{4} = 3\frac{\pi}{4}$

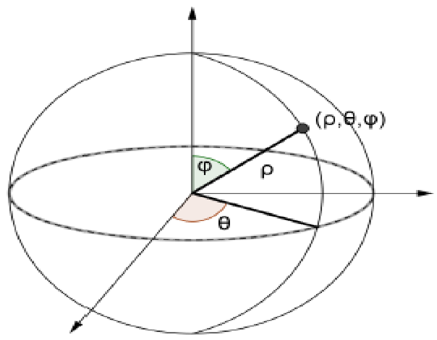
z= $\sqrt{2}$

Thus (-1 , 1 , $\sqrt{2}$) =($\sqrt{2}$, $3\frac{\pi}{4}$, $\sqrt{2}$)

Spherical Coordinate System

In the **spherical coordinate** system, a point P (x, y, z), whose Cartesian coordinates are (x, y, z) ; is described by an ordered triple (ρ , θ, ϕ).





- ϕ : Angle from positive z - axis to vector $\overrightarrow{OP}$.
- Where $0 \le \rho < \infty$; 0 ; $0 \le \theta \le 2\pi$; $0 < \phi < \pi$.

Note

To transformation from the Cartesian coordinates to the Spherical coordinates isgiven by

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\Phi = \cos^{-1}\left(\frac{z}{\rho}\right)$$

Example:

Convert the points from rectangular to spherical coordinates.

b) $(1,-1,-\sqrt{2})$
c) (0, 1,-1)
d) $(-1, 1, \sqrt{6})$

Solution

a)

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + -1^2 + -\sqrt{2}^2} = 2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$$

$\cos^{-1}(\frac{z}{\rho}) = \cos^{-1}(\frac{-\sqrt{2}}{2}) = -\frac{\pi}{4}$, $\Phi = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

the point is$(2, -\frac{\pi}{4}, \frac{3\pi}{4})$ in rectangular coordinates

b)

$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0^2 + -1^2 + -1^2} = 2$

$\theta = \tan^{-1}(\frac{y}{x}) = \tan^{-1}(\frac{1}{0}) = \frac{\pi}{2}$

$\cos^{-1}(\frac{z}{\rho}) = \cos^{-1}(\frac{-1}{2}) = \frac{\pi}{3}$, $\Phi = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

the point is$(2, \frac{\pi}{2}, \frac{2\pi}{3})$ in rectangular coordinates

c)

$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{-1^2 + 1^2 + \sqrt{6}^2} = 2\sqrt{2}$

$\theta = \tan^{-1}(\frac{y}{x}) = \tan^{-1}(\frac{1}{-1}) = -\frac{\pi}{4}$, $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\Phi = \cos^{-1}(\frac{z}{\rho}) = \cos^{-1}(\frac{\sqrt{6}}{2\sqrt{2}}) = \frac{\pi}{6}$

the point is$(2\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6})$ in rectangular coordinates

Note

To transformation from the Spherical coordinates to the Cartesian coordinates is given by

$x = \rho \sin \phi \cos \theta$

$y = \rho \sin \phi \sin \theta$

$z = \rho \cos \phi$

Example:

Convert the point $(4, \frac{\pi}{4}, \frac{\pi}{6})$ from spherical to rectangular coordinates.

Solution

$x = \rho \sin \phi \cos \theta = 4 \sin\frac{\pi}{6} \cos\frac{\pi}{4} = \sqrt{2}$

$y = \rho \sin \phi \sin \theta = 4 \sin\frac{\pi}{6} \sin\frac{\pi}{4} = \sqrt{2}$

$z = \rho \cos \phi = 4 \cos \frac{\pi}{6} = 2\sqrt{3}$

The point is$(\sqrt{2}, \sqrt{2}, 2\sqrt{3})$ in rectangular coordinates.

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand complex number

c. Central Ideas:

complex number

complex conjugate

Complex Arithmetic

square Roots of Complex Numbers.

d. Objectives: after the end of courses the student will be able to:

1-Find *complex conjugate*

2- find Complex Arithmetic

3- find square Roots of Complex Numbers.

Pre test

Q1: Solve the complex equation $m^2 - 7m + 9ni = n^2 i + 20i - 12$

Q2: Find the value of the following $(1+i)^{-8}$

Complex Numbers

Definitions.

Let $i2 = -1$.

$\therefore i = \sqrt{-1}$

Complex numbers are often denoted by *z*.

Just as R is the set of real numbers, C is the set of complex numbers.If *z* is a complex number, *z* is of the form

$z = x + iy \in$ C, for some $x, y \in$ R.

e.g. 3 + 4*i* is a complex number.

$z = x + iy$

↑ _

real part imaginary part.

If $z = x + iy$, $x, y \in$ R,

the real part of* z = _(z) = Re(z) = *x

***the imaginary part of* z = _(z) = Im(z) = *y*.**

eg. $z = 3 + 4i$

Re(z) = 3

Im(z)= 4.

If $z = x + iy$, then *z* ("*z bar*") is given by

$z = x - iy$

and is called the *complex conjugate* of *z*.

eg.If $z = 3+4i$, then $z = 3- 4i$.

Example. Solve $x2 - 2x + 3 = 0$.

complex Arithmetic.

Addition/Subraction.

Example 1. $(2 + 3i) + (4 + i) = 6 + 4i$.

Example 2. $(8 - 3i) - (-2 + 4i) = 10 - 7i$.

Multiplication/Division.

Example 1. $(2 + 3i)(1 + 2i) = 2 + 4i + 3i - 6 = -4 + 7i$

Example 2. $(3 - 2i)(3 + 2i) = 9 - (2i)^2 = 9 + 4 = 13$

∴ when we multiply two complex conjugates, we get a real number.

Example 3. 2+3*i*

1+4*i* = 2+3*i*

1+4*i*

× 1−4*i*

1−4*i* = (2+3*i*)(1−4*i*)

(1+4*i*)(1−4*i*) = 2−8*i*+3*i*−12*i*2

1−(4*i*)2 = 14−5*i*

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(*realising* the denominator)

Theorem. *If two complex numbers are equal then their real parts are equal and their imaginary parts are equal, i.e., if a+ib = c+id where a, b, c, d* ∈ R, *then a = c and b = d.*

Example 1. Find *x*, *y* if $(3 + 4i)^2 - 2(x - iy) = x + iy$.

Left hand side (LHS) = $9 - 16 + 24i - 2x + i2y$

$= -7 - 2x + i(24 + 2y)$

$\therefore\ -7 - 2x = x$

$3x = -7$

$x = -\frac{7}{3}$

& $24+2y = y$

$y = -24$ _

Solving simultaneously,

$6y = 60$

$y = 10$

& $\therefore\ x = -4.$ _

square Roots of Complex Numbers.

Example 1. Find the square root of $35 - 12i$.

Let

Example 2. Find the roots of $z^2 - (1 - i)z + 7i - 4 = 0$ in the form $a + ib$.

$z = \frac{(1 - i) \pm \sqrt{(1 - i)^2 - 4(1)(7i - 4)}}{2}$

$= \frac{(1 - i) \pm \sqrt{1 - 1 - 2i - 28i + 16}}{2}$

$= \frac{(1 - i) \pm \sqrt{16 - 30i}}{2}$

From beside,

$= \frac{(1 - i) \pm (5 - 3i)}{2}$

$= \frac{1 - i + 5 - 3i}{2}$ or $\frac{1 - i - (5 - 3i)}{2}$

$= 3 - 2i$ or $-2 + i.$ _

$\sqrt{16 - 30i} = (a + ib)$

$16 - 30i = a^2 - b^2 + i(2ab)$

$a^2 - b^2 = 16$

$2ab = -30$

$ab = -15$

$a = 5$ & $b = -3$

or $a = -5$ & $b = 3$

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand complex number

c. Central Ideas:

complex number

complex conjugate

Complex Arithmetic

square Roots of Complex Numbers.

d. Objectives: after the end of courses the student will be able to:

1-Find *complex conjugate*

2- find Complex Arithmetic

3- find square Roots of Complex Numbers.

Pre test

Q1: Solve the complex equation $m^2 - 7m + 9ni = n^2 i + 20i - 12$

Q2: Find the value of the following $(1+i)^{-8}$

Product of Complex Numbers.
The triangle *OQR* is constructed similar to _*AOP*. *A* is the point (1, 0).
Multiplication by *i*, −1, −*i*.
Multiplication by i, rotation 90◦ (anticlockwise).
Multiplication by −1, rotation 180◦ anticlockwise.
Multiplication by $-i$, rotation 270◦ anticlockwise
Geometric Representation of Locus Problems.
General forms:- $|z - z_1| = a$ represents a circle, centre at z_1 radius a units.
Example 1. $|z| = 1$.
Example 2. $|z - 3| = 2$.
Example 3. $|z - i| = 1$.
Example 4. $|z - 1 - 2i| = 2$
$|z - (1 + 2i)| = 2$ centre (1, 2), radius 2 units.
Example 5. $|z| \leq 3$ (note:- if less than, it is inside, if it is greater than, it is outside.)
Example 6. $2 < |z| \leq 3$.
Example 7. $|z| \leq 4$ and $0 \leq \arg z \leq \pi$
3 .
Example 8. $1 \leq$ _$(z) \leq 2$ if $z = x + iy$,
then _$(z) = y$ (& ∴ $1 \leq y \leq 2$)
Example 9. $-\pi$
6 $< \arg z \leq \pi$
3 .
Example 10. $1 \leq$ _$(z) \leq 2$ and _$(z) \leq -1$
Example 11. $1 \leq$ _$(z) \leq 2$ or _$(z) \leq -1$
Example 12. $|z| \leq 4$ or $0 \leq \arg z \leq \pi$
Using Algebra to Represent Locus Problems
Example 1. Show algebraically that $|z - 2 - i| = 4$ represents a circle with radius 4 units and centre (2, 1).
$|z - 2 - i| = 4$.
∴ $|x + iy - 2 - i| = 4$.
∴ $|(x - 2) + i(y - 1)| = 4$.
∴ _$(x - 2)^2 + (y - 1)^2 = 4$.
∴ $(x - 2)^2 + (y - 1)^2 = 16$.
which is a circle centre (2, 1), radius 4 units. _
Example 2. Sketch the curve: **(i)** _$(z^2) = 3$ **(ii)** _$(z^2) = 4$.
(i) _$(z^2) = 3$
_$((x + iy)^2) = 3$
_$(x^2 - y^2 + 2ixy) = 3$
$x^2 - y^2 = 3$.
(ii) _$(z^2) = 4$.
∴ $2xy = 4$.
∴ $xy = 2$.
Example 3. Describe in geometric terms, the curve described by $2|z| = z + z + 4$.
$2|z| = z + z + 4$.
∴ $2|x + iy| = x + iy + x - iy + 4$.
∴ 2_$x^2 + y^2 = 2x + 4 = 2(x + 2)$.
∴ _$x^2 + y^2 = x + 2$.
∴ $x^2 + y^2 = (x + 2)^2$.

$\therefore\ x^2 + y^2 = x^2 + 4x + 4.$
$\therefore\ y^2 = 4x + 4.$
$\Rightarrow$ sideways parabola at vertex $(-1, 0)$.
Example 4. Sketch the locus of _$(z + iz) < 2$.
_$(x + iy + i(x + iy)) < 2$.
$\therefore$ _$(x + iy + ix - y) < 2$.
$\therefore\ x - y < 2.$
Example 5. If $z_1 = 1+i$ & $z_2 = 2 + 3i$ find the locus of z if $|z - z_1| = |z - z_2|$.
$|x + iy - (1 + i)| = |x + iy - (2 + 3i)|.$
$\therefore\ |(x - 1) + i(y - 1)| = |(x - 2) + i(y - 3)|.$
_$(x - 1)^2 + (y - 1)^2 =$ _$(x - 2)^2 + (y - 3)^2.$
$(x - 1)^2 + (y - 1)^2 = (x - 2)^2 + (y - 3)^2.$
$x^2 - 2x+1+y^2 - 2y + 1 = x^2 - 4x+4+y^2 - 6y + 9.$
$\therefore\ 2x + 4y = 11.$
N.B. $|z - z_1| = |z - z_2|$ will always be a straight line. It will always be the perpendicular bisector of the interval joining z_1 to z_2.

($*$) **Note.** $\sin(A+B) = \sin A\cos B+\sin B\cos A$ & $\cos(A+B) = \cos A\cos B-\sin A\sin B$.
De Moivres Theorem. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$
Proof. (By mathematical induction for $n = 0, 1, 2, \ldots.$)
Step 1. Test $n = 0$.
L.H.S. $= (\cos\theta + i\sin\theta)^0$
$= 1$
R.H.S. $= \cos 0 + i\sin 0$
$= 1$
= L.H.S.
$\therefore$ it is true for $n = 0$.
Step 2. Assume true for $n = k$ i.e., $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta.$
Test for $n = k + 1$.
i.e., L.H.S. $= (\cos\theta + i\sin\theta)^{k+1}$ & R.H.S. $= \cos(k + 1)\theta + i\sin(k + 1)\theta$
$= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)^1$
$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$(since we have assumed it true for $n = k$)
$= \cos k\theta\cos\theta + i\sin\theta\cos k\theta + i\sin k\theta\cos\theta - \sin k\theta\sin\theta$
$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin\theta\cos k\theta + \sin k\theta\cos\theta)$
$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$ (see ($*$) above)
$= \cos(k + 1)\theta + i\sin(k + 1)\theta$
= R.H.S.
Step 3. If the result is true for $n = 0$, then true for $n = 0 + 1$, i.e., $n = 1$. If the result is true for $n = 1$, then true for $n = 1 + 1$, i.e., $n = 2$ ans so on for all nonnegative integers n _
Example 1. Simplify:
(a) $(\cos\theta - i\sin\theta)^{-4}$ **(b)** $(\sin\theta - i\cos\theta)^7$ **(c)** $\frac{(\cos 2\theta+i\sin 2\theta)^3}{(\cos\theta-i\sin\theta)^4}$.
(a) $(\cos\theta - i\sin\theta)^{-4} = \cos(-4\theta) - i\sin(-4\theta)$
$= \cos 4\theta + i\sin 4\theta$ _
(b) $(\sin\theta - i\cos\theta)^7 = (-i\cos\theta + \sin\theta)^7$
$= -i^7(\cos\theta - i\sin\theta)^7$
$= i(\cos 7\theta - i\sin 7\theta)$

$= \sin 7\theta + i\cos 7\theta$ _

(c) $\dfrac{(\cos 2\theta + i\sin 2\theta)^3}{(\cos\theta - i\sin\theta)^4} = \dfrac{(\cos\theta + i\sin\theta)^6}{(\cos\theta - i\sin\theta)^4}$

$= \dfrac{(\cos\theta + i\sin\theta)^6}{(\cos(-\theta) + i\sin(-\theta))^4}$

$= \dfrac{(\cos\theta + i\sin\theta)^6}{(\cos\theta + i\sin\theta)^{-4}}$

$= (\cos\theta + i\sin\theta)^{10}$

$= \cos 10\theta + i\sin 10\theta$ _

De Moivre's Theorem and the Argand Diagram

Example. If $z = \sqrt{3} + i$ represent the following on the Argand Diagram:

$z, iz, \frac{1}{z}, -z, 2z, z, z^2 + z, z^3 - z$

$z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

$z^{-1} = (2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}))^{-1}$

$= \frac{1}{2}(\cos-\frac{\pi}{6} + i\sin-\frac{\pi}{6})$

$= \frac{1}{2}(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})$

$2z = 4(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

$z^2 = (2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}))^2$

$= 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

$z^3 = (2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}))^3$

$= 8(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

Solution on next page.

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand A differential equation

c. Central Ideas:

- **First Ordinary Differential Equations**
- **Methods to solve the first order first degree differential equations**

d. Objectives: after the end of courses the student will be able to:

solve the first order first degree differential equations
Pre test
Q1: solve the first order first degree differential equations

$$\frac{x\,dy}{dx} + 3y = x^3$$

Differential Equations

A differential equation is an equation that involves one or more derivatives.
Differential equations are **classified by**:

1. Type: there are two type

A- **Ordinary differential equation:-** Equation which involve only one independent variable is called ordinary differential equation .

For example

1- $\frac{dy}{dx} = x + 5$

Is Ordinary differential equation, y is unknown function (dependent variable) and x is independent variable.

2- $y'' + x2(y'')^2 + y'$=cosx

Is Ordinary differential equation, is y unknown function (dependent variable) and x is independent variable.

B- **Partial differential equation**:-Equation which involve more than one independent variable called Partial differential equation

For example

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2z$$

Is partial.differential equation, is z unknown function (dependent variable) x and y is independent variable.

2. Order: The order of differential equation is the highest order derivative that occurs in the equation.

3. Degree: The exponent of the highest power of the highest order derivative.

For examples

Ex1:

$\frac{dy}{dx} = 5x + 3$ 1st order-1st degree

Ex2:

$\left(\frac{d^3 y}{dx^3}\right)^2 + \left(\frac{d^2 y}{dx}\right)^5$ 3rd order-2nd degree

Ex3:

$4\frac{d^3 y}{dx^3} + \sin x\frac{d^2 y}{dx^2} + 5xy = 0$ 3rd order-1st degree

H.W:

Exercise: Find the order and degree of these differential equations.

1. $\frac{dy}{dx} + \cos x = 0$ ans:1st order-1st degree
2. $3dx + 4y^2 dy = 0$ ans:1st order-1st degree
3. $\frac{d^2 y}{dx^2} + y = y^2$
4. $(y'')^2 + 2y' = x^2$
5. $y''' + 2(y'')^2 = xy$

Definition

The **solution** of the differential equation in the unknown function y and the independent variable x is a function y(x) that satisfies the differential equation.
i.e. any equation satisfying the differential equation is called solution of the differential equation

Exercise: Show that $y = a\cos 2x + b\sin 2x$ is a solution to y''+4y=0

First Ordinary Differential Equations

Ordinary Differential Equations are equation which involve only one independent variable

To solve the first order first degree differential equations we have the following cases

1- Variable Separable

First order differential equations. can be solved by integration if it is possible to collect all y terms with dy and all x terms with dx, that is, if it is possible to write the differential equations. in the form

$$f(x)dx + g(y)dy = 0$$

then the general solution is:

$$\int f(x)dx + \int g(y)dy = c \qquad \text{where } \mathbf{c} \text{ is an arbitrary constant.}$$

Ex 1:- Solve $xdy = ydx$

$$(ydx - xdy = 0)\frac{1}{xy}$$

$\frac{dx}{x} - \frac{dy}{y} = 0$ by integral of two sides

$$\int \frac{dx}{x} - \int \frac{dy}{y} = 0$$

$$Lnx - lny = lnc$$

$$Ln\frac{x}{y} = lnc$$

$$\frac{x}{y} = c$$

$$y = \frac{x}{c}$$

Ex2: Solve the D.E. $x(2y-3)dx + (x^2+1)dy = 0$

$$\frac{x}{(x^2+1)}dx + \frac{1}{(2y-3)}dy = 0$$

$$\int \frac{x}{(x^2+1)}dx + \int \frac{1}{(2y-3)}dy = 0$$

$$\frac{1}{2}\ln|(x^2+1)| + \frac{1}{2}\ln|(2y-3)| = \frac{1}{2}\ln c$$

Ex 3: –Solve the D.E $xe^y\, dy + \frac{x^2+1}{y}\, dx = 0$

$$\int ye^y\, dy + \int \frac{x^2+1}{x} dx = 0$$

$$\int ye^y\, dy + \int (x + \frac{1}{x}) dx = 0$$

$$ye^y - e^y + \left(\frac{x^2}{2} + \ln x\right) = c$$

y	e^y
1	e^y
0	e^y

H.W:

Exercise: Separate the variables and solve.

1- $x(2y-3)dx + (x^2+1)dy = 0$

2- $\frac{dy}{dx} = \frac{4y}{x(y-3)}$

3- $\frac{dy}{dx} = e^{x-y}$

4- $\sqrt{xy}\frac{dy}{dx} = 1$

2- Homogeneous

Definition : A function of f(x, y) is said to be **homogenous** of degree **n** if

$$f(kx, ky) = k^n f(x, y)$$

Where **k** is constant

For example

$f(x, y) = x^2 + 3xy + y^2, f(kx, ky) = k^2x^2 + 3kx.ky + k^2x^2$

$= k^2(x^2 + 3xy + x^2) = k^2 f(x, y)$

$\therefore f(x, y)$ is a homogenous function of degree two .

Now, When the differential equation as form

$$M(x, y)dx + N(x, y)dy = 0$$

Where M and N are function of x and y is called homogenous if satisfy the condition

$$M(kx, ky) = k^n M(x, y)$$
$$N(kx, ky) = k^n N(x, y)$$

Where k is constant

For example

1-$(x^2 - y^2)dx + 2xydy = 0$

$$M(x, y) = x^2 - y^2, N(x, y) = 2xy$$
$$M(kx, ky) = (kx)^2 - (ky)^2 = k^2\,(x^2 - y^2) = k^2\,M$$
$$N(kx, ky) = 2(kx)(ky) = 2k^2\,xy = k^2\,N$$

The equation is a homogenous

2-$(x - y)dx + xydy = 0$

$$M(x, y) = x - y, N(x, y) = xy$$
$$M(kx, ky) = kx - ky = k(x - y) = kM$$
$$N(kx, ky) = (kx)(ky) = k^2\,xy = k^2\,N$$

The equation is not homogenous

If the equation is **homogeneous** we can solved by the following method :-

Put in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad (1)$$

To solve it put $v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substitute in eq. (1) gives

$$v + x\frac{dv}{dx} = f(v) \Rightarrow f(v) - v = x\frac{dv}{dx}$$

$$\frac{dv}{f(v)-v} = \frac{dx}{x} \quad \text{(Separable D.E)}$$

Integration of both sides given the final solution

$$\int \frac{dv}{f(v) - v} = \ln|x| + c$$

Example

Solve $(x^2+y^2)dx + 2xydy = 0$

Solution

$M(x,y) = x^2+y^2, N(x,y) = 2xy$

$M(kx,ky) = k^2\,(x^2+y^2), N(kx,ky) = 2k^2\,(xy)$

$M(x,y)$ *and* $N(x,y)$ are hom.

$2xydy = -(x^2+y^2)dx$

$\frac{dy}{dx} = \frac{-(x^2+y^2)}{2xy}$ (1)

Let $y = vx$ *and* $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substitute y *and* $\frac{dy}{dx}$ in eq. (1) gives

$$v + \frac{xdv}{dx} = \frac{-(x^2+v^2x^2)}{2x^2v}$$

$$v + \frac{xdv}{dx} = \frac{-(1+v^2)}{2v}$$

$$\frac{xdv}{dx} = \frac{-(1+v^2)}{2v} - v$$

$$\frac{xdv}{dx} = \frac{-(1+v^2) - 2v^2}{2v}$$

$$\frac{xdv}{dx} = \frac{-(1+3v^2)}{2v}$$

$$\frac{x}{dx} = \frac{-(1+3v^2)}{2vdv}$$

$$\frac{dx}{x} = \frac{2vdv}{-(1+3v^2)}$$

$$\frac{dx}{x} + \frac{2vdv}{(1+3v^2)} = 0$$

Integration of both sides given the final solution

$$\ln|x| + \frac{1}{3}\ln|1+3v^2| = c$$

H.W:

Solve

1- $(x^3-3x^2y)dx - (x^3-x^3)dy = 0$

2- $\frac{dy}{dx} = \frac{x+y}{x-y}$

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Q1: solve the first order first degree differential equations

$$\frac{x\,dy}{dx} + 3y = x^3$$

3-Exact

A differential equation $M(x, y)dx + N(x, y)dy = 0$ is said to be **exact** if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

For example

1- The equation $(x^2 + y^2)dx + (2xy + cosy)dy$ isexact because the partial derivative $\frac{\partial M}{\partial y} = \frac{\partial(x^2+y^2)}{\partial y} = 2y, \frac{\partial N}{\partial x} = \frac{\partial(2xy+cosy)}{\partial x} = 2y$ are equal.

2- The equation $(x + 3y)dx + (x^2 + cosy)dy$ is not exact because the partial derivative $\frac{\partial M}{\partial y} = \frac{\partial(x+3y)}{\partial y} = 3, \frac{\partial N}{\partial x} = \frac{\partial(x^2+cosy)}{\partial x} = 2x$ are not equal.

Steps for solving an Exact differential equation

1- Match the equation to the form $M(x, y)dx + N(x, y)dy = 0$ to identify M and N.

2- Integrate M(or N) with respect to x (or y) , writing the constant of integration as g(y) or g(x) .

3- Differential with respect to y (or x) and set the result equal to N (or M) to find $g'(y) or\ g'(x)$.

4- Integral to find g(y) or g(x).

5- Write the solution of the exact equation as f(x, y)=c

Example

Solve $(x^2+y^2)dx + (2xy + cosy)dy = 0$

Solution

The equation $(x^2+y^2)dx + (2xy + cosy)dy$ is exact because

$\frac{\partial M}{\partial y} = \frac{\partial (x^2+y^2)}{\partial y} = 2y, \frac{\partial N}{\partial x} = \frac{\partial (2xy+cosy)}{\partial x} = 2y$ are equal.

Step1 Match the equation to the form $M(x,y)dx + N(x,y)dy = 0$ to identify M.

$$M(x,y) = x^2+y^2$$

Step2Integrate M with respect to x , writing the constant of integration as g(y) .

$$\int M(x,y)dx = \int (x^2+y^2)dx = \frac{x^3}{3} + xy^2 + g(y)$$

Step3 Differential with respect y and set the result
equal to N to find $g'(y)$.

$$\frac{\partial [\frac{x^3}{3}+xy^2+g(y)]}{\partial y} = 2xy+g'(y)$$

$$2xy+g'(y) = 2xy + cosy \Rightarrow g'(y) = cosy$$

Step4 Integral to find g(y).

$$\int g'(y)dy = \int cosydy = siny$$

Step5 Write the solution of the exact equation as f(x, y)=c

$$\frac{x^3}{3} + xy^2 + siny = c$$

H.W:

Solve the following equation

1- $\frac{dy}{dx} = \frac{x^3-3yx^2}{x^3-y^3}$

2- $(x^2-3y^2 + x + y - 2)dx + (x - 6xy + y^2 + 10)dy = 0$

4-First – order linear differential equation

A differential equation that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Where P and Q are function of the x is called **a linear first orde equation** .

The solution is

$$y = \frac{1}{I(x)}\left[\int I(x)Q(x)dx \ + c\right]$$

Where $I(x) = e^{\int P(x)dx}$

Steps for solving a linear order equation

1- Put it in standard form and identify the functions P(x) and Q(x).

2- Find an integral of p(x) i.e $\int p(x)dx$

3- Find the integrating factor $I(x) = e^{\int P(x)dx}$

4- Find y using the following equation

$y = \frac{1}{I(x)}[\int I(x)Q(x)dx \ + c]$, where c is $constant$

Example

Solve $x\frac{dy}{dx} - 3y = x^2$

Solution

Step1 Put the equation in standard form and identify the functions P(x) and Q(x).To do so ,we divide both sides of the equation by the coefficient of $\frac{dy}{dx}$, In this case x , obtaining

$$\frac{dy}{dx} - \frac{3}{x}y = x$$

$$p(x) = \frac{-3}{x} \; and \; Q(x) = x$$

Step2 Find an integral of p(x)

$$\int p(x)dx = \int \frac{-3}{x}dx = -3\ln x$$

Step3 Find the integrating factor $I(x)$

$$I(x) = e^{\int P(x)dx} = e^{-3lnx} = e^{lnx^{-3}} = \frac{1}{x^3}$$

Step4 Find the soluation

$$y = \frac{1}{I(x)}[\int I(x)Q(x)dx + c] \text{, where c is } constant$$

$$y = \frac{1}{1/x^3}\left[\int \frac{1}{x^3}xdx + c\right]$$

The solution is the function $y = x^3\left[\frac{-1}{x} + c\right] = cx^3 - x^2$

H.W:

Solve the following equations

1- $x\frac{dy}{dx} + 3y = x^2$
2- $(1 + x^2)dy + (y - \tan^{-1}x)dx = 0$
3- $\frac{dy}{dx} - y\tan x = 1$

5- The Bernoulli Equation

The equation $\frac{dy}{dx} + p(x)y = Q(x)y^n$ ---(1)

if $n \neq 0$ called Bernoulli equation. We shall show transform this equation to linear equation. In fact we must reduce this equation to linear , product eq.(1) by y^{-n}

$$\left[\frac{dy}{dx} + p(x)y = Q(x)y^n\right] \ y^{-n}$$

$$\frac{dy}{dx}y^{-n} + p(x)y^{1-n} = Q(x) \qquad \text{----- (2)}$$

Let

$$w = y^{1-n}$$

$$dw = (1-n)y^{1-n-1}\,dy$$

Or

$$\frac{dw}{1-n} = y^{-n}\ dy \quad \text{Put in (2)}$$

$$\frac{dw}{(1-n)dx} + p(x)w = Q(x)$$

Or

$$\frac{dw}{dx} + (1-n)p(x)w = (1-n)Q(x)$$

Example

Solve the following differential equation

$$\frac{dy}{dx} + \frac{y}{x} = y^2$$

Solution

$$\left[\frac{dy}{dx} + \frac{y}{x} = y^2\right] y^{-2}$$

$$\frac{dy}{dx}y^{-2} + \frac{y^{-1}}{x} = 1 \qquad \text{----(1)}$$

Let $w = y^{-1} \Rightarrow dw = -y^{-2}dy$

$-dw = y^{-2}dy$ put in eq.(1)

$$-\frac{dw}{dx}+\frac{w}{x}=1$$

$$\frac{dw}{dx}-\frac{w}{x}=-1$$

$$p=-\frac{1}{x}, Q=-1$$

$$I=e^{\int p dx}=e^{\int -\frac{dx}{x}}=e^{-lnx}=\frac{1}{x}$$

The solution

$$Iw=\int IQdx+c$$

$$\frac{w}{x}=\int \frac{-1}{x}dx+c=-\ln x+c$$

$$\frac{1}{xy}=-\ln x+c$$

$$y=\frac{1}{x(c-\ln x)}$$

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand A differential equation

c. Central Ideas:

- **Second Ordinary Differential Equations**
- **Methods to solve the second order linear homogeneous differential equations**

d. Objectives: after the end of courses the student will be able to:

- **solve the second order linear homogeneous differential equations**

pre test
Q1: solve the first order first degree differential equations

$$y''- y' - 2y = 0$$

Second Order Linear Homogeneous Equation

The linear equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1n}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = f(x)$$

If $f(x) = 0$ then it is called homogeneous ; otherwise it is called non-homogeneous

<u>Linear Differenential operator</u>

It is convenient to introduce the symbol D to respect the operation of differenential with respect to x. That is, we write $Df(x)$ to mean $\frac{df}{dx}$. Furthermore, we define power of D to mean taking successive derivative:

$$D^2 f(x) = D\{Df(x)\} = \frac{d^2 f}{dx^2}, D^3 f(x) = D\{D^2 f(x)\} = \frac{d^3 f}{dx^3}$$

$$(D^2 + D - 2)f(x) = D^2 f(x) + Df(x) - 2f(x) = \frac{d^2 f}{dx^2} + \frac{df}{dx} - 2f(x)$$

<u>The Characteristic Equation</u>

The linear second order equation with constant real- number coefficient is

$$\frac{d^2 y}{dx^2} + 2a \frac{dy}{dx} + by = 0$$

Or, in operator notation

$$(D^2 + 2aD + b)y = 0$$

$$(D - r_1)(D - r_2)y = 0$$

Solution of $\frac{d^2 y}{dx^2} + 2a \frac{dy}{dx} + by = 0$ depended on root r_1 and r_2

Root r_1 and r_2	**Solution**
Real and unequal	$y = c_1 e^{r_1 x} + c_2 e^{r_{21} x}$

Real and equal	$y = (c_1 x + c_2)e^{r_1 x}$
Complex conjugate $a \pm ib$	$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

Example: solve the following equations

a) $\acute{\acute{y}} + \acute{y} - 2y = 0$
b) $\acute{\acute{y}} + 4\acute{y} + 4y = 0$
c) $\acute{\acute{y}} + 4\acute{y} + 6y = 0$
d) $\acute{\acute{y}} + 4y = 0$

Solution

1. $\acute{\acute{y}} + \acute{y} - 2y = 0$

The characteristic equation is

$$D^2 + D - 2 = 0$$

$$(D-1)(D+2) = 0$$

$$r_1 = 1 \; and \; r_2 = -2$$

The solution is

$$y = c_1 e^x + c_2 e^{-2x}$$

$b)\acute{\acute{y}} + 4\acute{y} + 4y = 0$

The characteristic equation is

$$D^2 + 4D + 4 = 0$$

$$(D+2)(D+2) = 0$$

The solution is

$$y = (c_1 x + c_2)e^{-2x}$$

C) $\acute{\acute{y}} + 4\acute{y} + 6y = 0$

The characteristic equation is

$$D^2 + 4D + 6 = 0$$

$$\mathrm{r}_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r_{1,2} = \frac{-4 \pm \sqrt{16 - 24}}{2}$$

$$r_{1,2} = \frac{-4 \pm \sqrt{-8}}{2}$$

$$\mathrm{r}_{1,2} = -2 \pm \sqrt{2}i$$

The solution is

$$y = e^{-2x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x)$$

d) $\acute{\acute{y}} + 4y = 0$

The characteristic equation is

$$D^2 + 4 = 0$$

$$(D - 2i)(D + 2i) = 0$$

$$r_1 = 2i$$

$$\boldsymbol{r_2} = -2i$$

The solution is

$$y = (c_1 \cos 2x + c_2 \sin 2x)$$

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand A differential equation

c. Central Ideas:

- **Second order Non-homogeneous Linear Equations**
- **Methods to solve Second order Non-homogeneous Linear Equations**

d. Objectives: after the end of courses the student will be able to:

- **SolveSecond order Non-homogeneous Linear Equations**

Pre test
Q1: solve differential equations
$\boldsymbol{y''- y' - 2y = 4x^3}$ [use undetermined coefficient method]

Second order Non-homogeneous Linear Equations

Now, we solve non-homogeneous equations of the form

$$\frac{d^2y}{dx^2}+2a\frac{dy}{dx} + by = f(x)$$

The procedure has three basic steps.

First: we find the homogeneous solution y_h (h stand for homogeneous) of the reduced equation.

$$\frac{d^2y}{dx^2}+2a\frac{dy}{dx} + by = 0$$

Second: we find a particular solution y_p of the complete equation.

Finally: we add y_p to y_h to form the general solution of the complete equation. So, the final solution is

$$y = y_h + y_p$$

Methods to find the particular solution y_p

1) **Variation of parameters:**

This method assumes we already know the homogeneous solution

$$y_h = c_1 u_1(x) + c_2 u_2(x)$$

the method consists of replacing the constants c_1 and c_2 by function $v_1(x)$ and $v_2(x)$, then requiring that the new expression

$$y_h = v_1 u_1 + v_2 u_2$$

and by solving the following two equations

$$\acute{v}_1 u_1 + \acute{v}_2 u_2 = 0$$

$$\acute{v}_1 \acute{u}_1 + \acute{v}_2 \acute{u}_2 = f(x)$$

for the un know function $\acute{v}_1$ and $\acute{v}_2$ using the following matrix notation

$$\begin{bmatrix} u_1 & u_2 \\ \acute{u}_1 & \acute{u}_2 \end{bmatrix} \begin{bmatrix} \acute{v}_1 \\ \acute{v}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

Finally v_1 and v_2 can be found by integration.

In appling the method of Variation of parameters to find the particular solution, **the following steps are taken:**

i. Find $\acute{v}_1$ and $\acute{v}_2$ using the following equation

$$\acute{v}_1 = \frac{\begin{vmatrix} 0 & u_2 \\ f(x) & \acute{u}_2 \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ \acute{u}_1 & \acute{u}_2 \end{vmatrix}} = \frac{-u_2 f(x)}{u_1 \acute{u}_2 - u_2 \acute{u}_1}$$

$$\acute{v}_2 = \frac{\begin{vmatrix} u_1 & 0 \\ \acute{u}_1 & f(x) \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ \acute{u}_1 & \acute{u}_2 \end{vmatrix}} = \frac{u_1 f(x)}{u_1 \acute{u}_2 - u_2 \acute{u}_1}$$

ii. Integrate $\acute{v}_1$ and $\acute{v}_2$ to find v_1 and v_2.

iii. Write the particular solution

$$y_p = v_1 u_1 + v_2 u_2$$

Example: solve the equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-3y=6$$

Solution: the homogeneous solution y_h can be found using the reduced equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-3y=0$$

The characteristic equation is $D^2+2D-3=0$ and the roots of this equation are $r_1=-3$ and $r_2=1$, so

$$y_h=c_1e^{-3x}+c_2e^{x}\Rightarrow u_1=e^{-3x}\ and\ u_2=e^{x}$$

$$\acute{v}_1e^{-3x}+\acute{v}_2e^{x}\ =0$$

$$-3\acute{v}_1e^{-3x}+\acute{v}_2e^{x}\ =6$$

$$\acute{v}_1=\frac{\begin{vmatrix}0 & e^{x}\\ 6 & e^{x}\end{vmatrix}}{\begin{vmatrix}e^{-3x} & e^{x}\\ -3e^{-3x} & e^{x}\end{vmatrix}}=\frac{-6e^{x}}{e^{-3x}e^{x}-e^{x}(-3e^{-3x})}=\frac{-6e^{x}}{e^{-2x}+3e^{-2x}}=\frac{-6e^{x}}{4e^{-2x}}$$

$$\acute{v}_1=\frac{-3}{2}e^{3x}$$

$$\acute{v}_2=\frac{\begin{vmatrix}e^{-3x} & 0\\ -3e^{-3x} & 6\end{vmatrix}}{\begin{vmatrix}e^{-3x} & e^{x}\\ -3e^{-3x} & e^{x}\end{vmatrix}}=\frac{6e^{-3x}}{4e^{-2x}}=\frac{3}{2}e^{-x}$$

$$v_1=\int\frac{-3}{2}e^{3x}dx=\frac{-1}{2}e^{3x}$$

$$v_2=\int\frac{3}{2}e^{-x}dx=\frac{-3}{2}e^{-x}$$

$$y_p=v_1u_1+v_2u_2$$

$$=\left(\frac{-1}{2}e^{3x}\right)e^{-3x}+\left(\frac{-3}{2}e^{-x}\right)e^{x}=-2$$

The general solution is

$$y=y_h+y_p$$

$$=c_1e^{-3x}+c_2e^{x}-2$$

Example: solve the equation

$$\acute{\acute{y}} - 2y' + 1 = e^x lnx$$

solution

the characteristic equation is $D^2 - 2D + 1 = 0 \Longrightarrow (D-1)(D-1) = 0$

the roots are $r_1 = r_2 = 1$

the solution is $y_h = (c_1 x + c_2)e^x = c_1 x e^x + c_2 e^x$

from that we have $u_1(x) = xe^x \ and \ u_2(x) = e^x$

$$u_1 \acute{v}_1 + u_2 \acute{v}_2 = 0$$

$$\acute{u}_1 \acute{v}_1 + \acute{u}_2 \acute{v}_2 = f(x)$$

$$xe^x \acute{v}_1 + e^x \acute{v}_2 = 0$$

$$(xe^x + e^x)\acute{v}_1 + e^x \acute{v}_2 = e^x lnx$$

Let $M = \begin{vmatrix} xe^x & e^x \\ xe^x + e^x & e^x \end{vmatrix} = xe^{2x} - (xe^{2x} + e^{2x}) = -e^{2x}$

$$\acute{v}_1 = \frac{\begin{vmatrix} 0 & e^x \\ e^x lnx & e^x \end{vmatrix}}{M} = \frac{-lnx\, e^{2x}}{-e^{2x}} = lnx$$

$$\acute{v}_2 = \frac{\begin{vmatrix} xe^x & 0 \\ xe^x + e^x & e^x lnx \end{vmatrix}}{M} = \frac{xlnx\, .\, e^{2x}}{-e^{2x}} = -xlnx$$

$$v_1 = \int lnx\, dx$$

Let $u = lnx\, ,\, dv = dx \Longrightarrow du = \frac{1}{x}dx, v = x$

$$v_1 = xlnx - \int x.\frac{1}{x}dx = xlnx - x$$

$$v_2 = -\int xlnxdx$$

Let $u = lnx \Longrightarrow du = \frac{1}{x}dx, dv = xdx \Longrightarrow v = \frac{x^2}{2}$

$$v_2 = -\left(\frac{x^2}{2}lnx - \int \frac{x^2}{2}.\frac{1}{x}dx\right)$$

$$= -\left(\frac{x^2}{2}lnx - \int \frac{x}{2}dx\right)$$

$$= -\left(\frac{x^2}{2}lnx - \frac{x^2}{4}\right)$$

$$= \frac{x^2}{4} - \frac{x^2}{2}lnx$$

the particular solution is

$$y_p = v_1u_1 + v_2u_2$$

$$= (xlnx - x)xe^x + \left(\frac{x^2}{4} - \frac{x^2}{2}lnx\right)e^x$$

$$= x^2e^xlnx - x^2e^x + \frac{x^2}{4}e^x - \frac{x^2}{2}e^xlnx$$

$$= \frac{x^2}{2}e^xlnx - \frac{3x^2}{4}e^x$$

The complete solution is

$$y = y_h + y_p$$

$$= c_1xe^x + c_2e^x + \frac{x^2}{2}e^xlnx - \frac{3x^2}{4}e^x$$

2) **Undetermined coefficients:** this method gives us the particular solution for selected equations.

The method of undetermined coefficients for selected equation of the form

$$\frac{d^2y}{dx^2}+2a\frac{dy}{dx} + by = f(x)$$

If $f(x)$ **has a term of**	**the expression for y_p**
e^{rx}	$A\,e^{rx}$
$\sin(kx)\,,\cos(kx)$	$\mathrm{B}\cos(kx) + C\,\sin(kx)$
$ax^2 + bx + c$	$Dx^2 + Ex + F$

Important Note: this expression used for y_p should not have any term similar to the terms of the y_h. Otherwise, multiplying the term that is similar to y_h repeating by x until it becomes different.

Example: solve the equation

1. $\acute{\acute{y}} - 6\acute{y} + 9y = e^{3x}$
2. $\acute{\acute{y}} - \acute{y} = 5e^{x} - \sin(2x)$
3. $\acute{\acute{y}} - \acute{y} - 2y = 4x^3$

Solution: 1) The homogeneous solution y_h can be found using the reduced equation

$$\acute{\acute{y}} - 6\acute{y} + 9y = 0$$

the characteristic equation is

$$D^2 - 6D + 9 = 0$$
$$(D-3)^2 = 0$$

the roots are $r_1 = r_2 = 3$

$$y_h = (c_1 x + c_2)e^{3x}$$

Since $f(x) = e^{3x}$ then let $y_p = Ae^{3x}$. But, Ae^{3x} is similar the second term of the y_h. So, let $y_p = Axe^{3x}$. Again Axe^{3x} is also similar to the first term of the y_h. Finally, let $y_p = Ax^2e^{3x} \Longrightarrow \acute{y}_p = 3Ax^2e^{3x} + 2Axe^{3x}$

$$\acute{\acute{y}}_p = (9x^2e^{3x} + 6xe^{3x}) + (6Axe^{3x} + 2Ae^{3x})$$

$$= 9Ax^2e^{3x} + 12xe^{3x} + 2Ae^{3x}$$

substituting into the differential equation $\acute{\acute{y}} - 6\acute{y} + 9y = e^{3x}$

we get

$(9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}) - 6(3Ax^2e^{3x} + 2Axe^{3x}) + 9Ax^2e^{3x} = e^{3x}$

$\Longrightarrow 2Ae^{3x} = e^{3x} \Longrightarrow 2A = 1 \Longrightarrow A = \frac{1}{2}$

$\therefore y_p = \frac{1}{2}x^2e^{3x}$

the general solution is

$$y = (c_1x + c_2)e^{3x} + \frac{1}{2}x^2e^{3x}$$

2) The homogeneous solution y_h can be found using the reduced equation

$$\acute{\acute{y}} - \acute{y} = 0$$

the characteristic equation is

$$D^2 - D = 0 \Longrightarrow D(D-1) = 0$$

the roots are $r_1 = 1$ and $r_2 = 0$

$$\therefore\ y_h = c_1e^x + c_2$$

Since $f(x) = 5e^x - sin2x$ then let $y_p = Ae^x + Bcos(2x) + Csin(2x)$. But, Ae^x is similar to the first term of the homogeneous solution. So, let

$$y_p = Axe^x + Bcos(2x) + Csin(2x)$$

$$\acute{y}_p = Axe^x + Ae^x - 2Bsin(2x) + 2Ccos(2x)$$

$$\acute{\acute{y}}_p = Axe^x + Ae^x + Ae^x - 4Bcos(2x) - 4Csin(2x)$$

substituting into the differential equation $\acute{\acute{y}} - \acute{y} = 5e^x - \sin(2x)$

we get

$$Axe^x + 2Ae^x - 4Bcos(2x) - 4Csin(2x) - \left(Axe^x + Ae^x - 2Bsin(2x) + 2Ccos(2x)\right) = 5e^x - \sin(2x)$$

$$Ae^x - (4B + 2C)cos2x + (2B - 4C)sin2x = 5e^x - \sin 2x$$

$$A = 5, (4B + 2C) = 0, (2B - 4C) = -1$$

$$A = 5, B = -\frac{1}{10}, C = \frac{1}{5}$$

So,

$$y_p = 5xe^x - \frac{1}{10}cos2x + \frac{1}{5}sin2x$$

the general solution is

$$y = y_h + y_p$$

$$y = c_1e^x + c_2 + 5xe^x - \frac{1}{10}cos2x + \frac{1}{5}sin2x$$

3) The homogeneous solution y_h can be found using the reduced equation

$$\acute{\acute{y}} - \acute{y} - 2y = 0$$

the characteristic equation is

$$D^2 - D - 2 = 0$$

$$(D-2)(D+1) = 0$$

the roots are $r_1 = 2$ and $r_2 = -1$

$$y_h = c_1e^{2x} + c_2e^{-x}$$

Since $f(x) = 4x^3$ then let

$$y_p = Ax^3 + Bx^2 + Cx + D.$$

$$\acute{y}_p = 3Ax^2 + 2Bx + C$$

$$\acute{\acute{y}}_p = 6Ax + 2B$$

substituting in to differential equation $\acute{\acute{y}} - \acute{y} - 2y = 4x^3$

we get

$$6Ax + 2B - (3Ax^2 + 2Bx + C) - 2(Ax^3 + Bx^2 + Cx + D) = 4x^3$$

$$-2Ax^3 - (3A + 2B)x^2 + (6A - 2B - 2C)x + (2B - C - 2D) = 4x^3$$

$$-2A = 4 \Longrightarrow A = -2$$

$$3A + 2B = 0 \Longrightarrow 3(-2) + 2B = 0 \Longrightarrow 2B = 6 \Longrightarrow B = 3$$

$$6A - 2B - 2C = 0 \Rightarrow 6(-2) - 2(3) - 2C = 0 \Rightarrow C = -9$$

$$2B - C - 2D = 0 \Rightarrow 2(3) - (-9) - 2D = 0 \Rightarrow D = \frac{15}{2}$$

So, $y_p = -2x^3 + 3x^2 - 9x + 7.5$

the general solution is

$$y = c_1e^{2x} + c_2e^{-x} - 2x^3 + 3x^2 - 9x + 7.5$$

Application of Differential Equation

Example: For the circuit shown below. Find expression for the current $i(t)$ if $V_s(t) = sinwt$.

Solution:

$$V_s = Ri + L\frac{di}{dt}$$

$$sinwt = 8i + 0.1\frac{di}{dt} \Rightarrow \frac{di}{dt} + 80i = 10\ sinwt\ (linear\ O.D.E.)$$

$P(t) = 80$ and $Q(t) = 10\ sinwt$

$$I(t) = e^{\int P(t)dt} = e^{\int 80dt} = e^{80t}$$

now

$$I(t).i = \int I(t)Q(t)dt + c$$

$$e^{80t}.i = \int e^{80t}.10\ sinwtdt + c$$

$$e^{80t}.i = 10\int e^{80t}.sinwtdt + c$$

$$e^{80t}.i = 10\left[\frac{e^{80t}}{(80)^2 + w^2}.(80sinwt - wcoswt)\right] + c$$

$$i = \frac{10}{(80)^2 + w^2}.(80sinwt - wcoswt) + ce^{-80t}$$

Example: For the circuit shown below. Find the current $i(t)$ if $i(0) = 1A$.

Solution:

$$Ri + \frac{1}{c}\int idt = V_s$$

$$R\frac{di}{dt} + \frac{1}{c}i = \frac{d}{dt}V_s$$

$$10\frac{di}{dt} + \frac{1}{500}i = 2 * 5cos5t$$

$$\frac{di}{dt} + 200i = cos5t$$

Using Linear O.D.E $\Longrightarrow$ $P(t) = 200, Q(t) = cos5t$

$I(t) = e^{\int P(t)dt} = e^{\int 200dt} = e^{200t}$

$$i.I(t) = \int I(t)Q(t)dt + c$$

$$i.e^{200t} = \int e^{200t}cos5tdt + c$$

$$i.e^{200t} = \frac{e^{200t}}{(200)^2 + 25}(200cos5t + 5sin5t) + c$$

$$i = \frac{1}{(200)^2 + 25}(200cos5t + 5sin5t) + c * e^{-200t}$$

$$\because i(0) = 1A$$

$$\therefore 1 = \frac{1}{(200)^2 + 25}(200 * 1 + 5 * 0) + c \Longrightarrow c = 1 - 4.997 * 10^{-3}$$

$$\Longrightarrow c = 0.998$$

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand Multiple Integrals

c. Central Ideas:

- **Double integral over Rectangular Region**

d. Objectives: after the end of courses the student will be able to:

solve Double integral over Rectangular Region

-

pre test

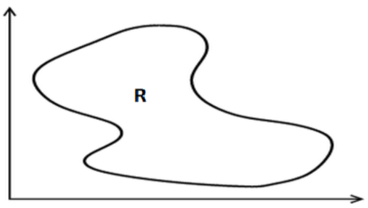
Q1 Evaluate $\int_0^1 \int_0^1 (x^2 + y^2)\, dxdy$

Multiple Integrals

Double integral

Let f(x,y) be a continuous function in side and on the boundary R, then

$\iint_R f(x,y)dA$ is called double integral of a function f(x,y) over R .



To evaluate the double integral:

(1) Double integral over Rectangular Region

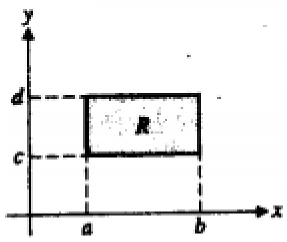
The double integral of a function f(x,y) over rectangle R where

R = {(x,y) | $a \le x \le b$, $c \le y \le d$ }

is

$$\iint f(x,y)dA = \int_a^b [\int_c^d f(x,y)\ dy]\ dx$$

$$= \int_c^d [\int_a^b f(x,y)\ dx]\ dy$$



- R is called the region of integration.
- The expression *dA* indicates that this is an integral over a two dimensional region

Note

a) $\int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$

b) $\int_c^d \int_a^b f_1(x)f_2(y)dxdy = \int_a^b f_1(x)dx \int_c^d f_2(y)dy$

Example:

Evaluate $\iint (x+y)dA$, over R = {(x,y) | $1 \le x \le 3$, $-1 \le y \le 2$ }

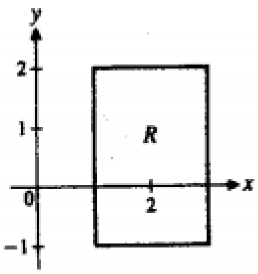
Solution

$\iint (x+y)dA = \int_1^3 \int_{-1}^2 (x+y)dydx$

$=\int_1^3 [(xy+\frac{y^2}{2}) \mid_{-1}^{2}]dx$

$=\int_1^3 [(2x+2)-(-x+\frac{1}{2})]dx$

$=\frac{3x^2}{2}+\frac{3x}{2} \mid_1^3 = 15$



$= [3\frac{x^2}{2} + \frac{3}{2}x \,|\, ^3_1 = (\frac{27}{2} + \frac{9}{2}) - (\frac{3}{2} + \frac{3}{2}) = 18 - 3 = 15$

2. With x integration first

$$\iint (x+y)dA = \int_{-1}^{2}\int_{1}^{3}(x+y)dxdy$$

$$= \int_{-1}^{2}[(\frac{x^2}{2} + yx) \,|\, ^3_1]\, dy$$

Example:

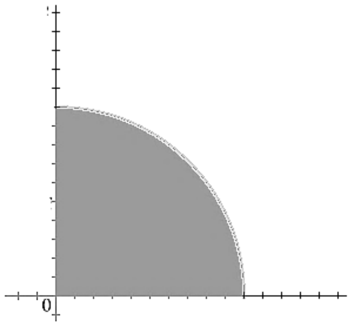
Evaluate $\int_0^1 \int_0^1 (x^2 + y^2)\, dxdy$

Solution

1. With x integration first

$$\int_0^1 \int_0^1 (x^2 + y^2)\, dxdy = \int_0^1 \left(\frac{x^3}{3} + xy^2\right)\Big|_1^0 dy$$

$$= \int_0^1 \left(\frac{1}{3} + y^2\right) dy = (\frac{1}{3}y + \frac{y^3}{3})\Big|_0^1 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$



2. With y integration first:

$$\int_0^1 \int_0^1 (x^2 + y^2)\, dydx$$

$$= \int_0^1 \left[x^2y + \frac{y^3}{3}\right]\Big|_0^1 dx = \int_0^1 \left(x^2 + \frac{1}{3}\right) dx$$

$$= (\frac{x^3}{3} + \frac{1}{3}x)\Big|_0^1 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Example:-

Evaluate $\int_1^3 \int_0^2 e^{2x+y}\, dydx$

Solution

1. With y integration first:

$$\int_1^3 \int_0^2 e^{2x}e^{y}dydx = \int_1^3 [e^{2x}e^{y}|_0^2 dx$$

$$= \int_1^3 [e^{2x}(e^2 - 1)]dx = (e^2 - 1)\int_1^3 e^{2x}\, dx$$

$$= (e^2 - 1)\,[\frac{1}{2}e^{2x}\Big|_1^3 = (e^2 - 1) * \frac{1}{2}(e^6 - e^2)$$

2. With x integration first

$$\int_0^2 \int_1^3 e^{2x}e^{y}\, dxdy = \int_0^2 \left[e^{y}\cdot\frac{1}{2}\, e^{2x}\right]_1^3 dy$$

$$= \int_0^2 \frac{e^{y}}{2}[e^6 - e^2]dy = \left(\frac{e^6 - e^2}{2}\right)\int_0^2 e^{y}\, dy$$

$$= \left(\frac{e^6 - e^2}{2}\right)[e^{y}|_0^2 = \left(\frac{e^6 - e^2}{2}\right) \times (e^2 - 1)$$

Example:-

Evaluate $\int_2^4 \int_1^2 6xy^2 dydx$ **.**

Solution

1. With y integration first

$$\int_2^4 [2xy^3 |_1^2 \, dx = \int_2^4 [16x - 2x]dx$$

$$= \int_2^4 14xdx = [7x^2 |_2^4 = 112 - 28 = 84$$

2. With x integration first

$$\int_1^2 \int_2^4 6xy^2 \, dxdy = \int_1^2 [3x^2y^2 |_2^4 dy$$

$$= \int_1^2 [48y^2 - 12y^2]dy = \int_1^2 36\, y^2 dy = [12y^3|_1^2 = 96 - 12 = 84$$

Example:-

Evaluate $\int_1^2 \int_0^1 \frac{1}{(2x+3y)^2} dxdy$

Solution

1. With x integration first

$$\int_1^2 \int_0^1 (2x+3y)^{-2} dxdy = \int_1^2 [\frac{-1}{2}(2x+3y)^{-1}\Big|_0^1 \; dy$$

$$= -\frac{1}{2}\int_1^2 \left[\frac{1}{2+3y} - \frac{1}{3y}\right] dy = -\frac{1}{2}\int_1^2 \left[\frac{1}{3}\frac{3}{2+3y} - \frac{1}{3}\frac{3}{3y}\right] dy$$

$$= -\frac{1}{6}[ln(2+3y) - ln(3y)]_1^2 = -\frac{1}{6}[(ln8 - ln6) - (ln5 - ln3)]$$

$$= -\frac{1}{6}[ln8 - ln6 - ln5 + ln3] = -\frac{1}{6}[ln8 - ln5 - ln2]$$

2. With y integration first

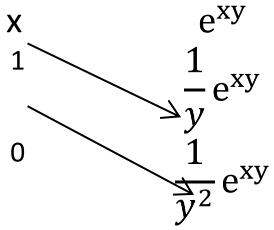
$$\int_0^1 \int_1^2 (2x+3y)^{-2} dydx = \int_0^1 [-\frac{1}{3}(2x+3y)^{-1}\Big|_1^2 \; dx$$

$$= -\frac{1}{3}\int_0^1 \left[\frac{1}{2x+6} - \frac{1}{2x+3}\right] dx = -\frac{1}{3}\left[\frac{1}{2}ln(2x+6) - \frac{1}{2}ln(2x+3)\right]_0^1$$

$$= -\frac{1}{6}[(ln8 - ln5) - (ln6 - ln3)] = -\frac{1}{6}[ln8 - ln5 - ln2]$$

Example:-

Evaluate $\int_{0}^{1}\int_{-1}^{2} xe^{xy}dxdy$



Solution

With x integration first

$$\int_{0}^{1}\int_{-1}^{2} xe^{xy}dxdy = \int_{0}^{1}\left[\frac{x}{y}e^{xy}-\frac{1}{y^2}e^{xy}\right]_{-1}^{2}dy$$

$$= \int_{0}^{1}\left[\left(\frac{2}{y}e^{2y}-\frac{1}{y^2}e^{2y}\right)-\left(\frac{-1}{y}e^{-y}-\frac{1}{y^2}e^{-y}\right)\right]dy$$

We are not even going to continue here as these are very difficult integrals to do ,
Then complete integration solution with y first .

$$\int_{-1}^{2}\int_{0}^{1} xe^{xy}\,dydx = \int_{-1}^{2}[e^{xy}]_{0}^{1}dx$$

$$= \int_{-1}^{2}[e^{x}-1]dx = [e^{x}-x]|_{-1}^{2}$$

$$= (e^{2}-2)-(e^{-1}+1) = e^{2}-e^{-1}-3$$

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand Multiple Integrals

c. Central Ideas:

Double Integrals for Bounded Non Rectangular Regions

d. Objectives: after the end of courses the student will be able to:

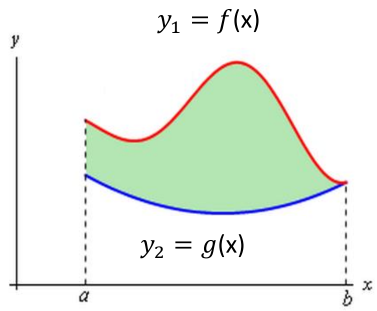
solve Double Integrals for Bounded Non Rectangular Regions

pre test

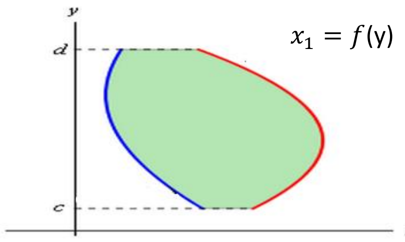
Q1 Evaluate $\int_1^2 \int_y^{y^3} e^{\frac{x}{y}} dx dy$

-Double Integrals for Bounded Non Rectangular Regions

a. $\int\int f(x,y)dA = \int_a^b \int_{g(x)}^{f(x)} f(x,y)\, dydx$



b. $\int\int f(x,y)\, dA = \int_c^d \int_{g(x)}^{f(x)} f(x,y)\ dxdy$



Note

- **If the limits of integration are constant, the region is rectangular.**
- **If the limits of integration are not constant, the region is non-rectangular.**

Example:-

Integrate the function f(x,y) = x^2y over the region on bounded by

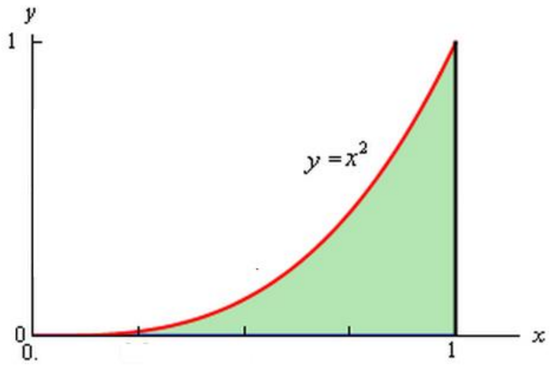
y = x^2, x =0, x =1 y =0

Solution

1. With y integration first

$$\int_0^1 \int_0^{x^2} x^2 ydydx = \int_0^1 x^2 [\frac{y^2}{2}\Big|_0^{x^2} dx$$

$$= \int_0^1 \frac{x^6}{2}\, dx = \frac{x^7}{14}\Big|_0^1 = \frac{1}{14}$$



2. With x integration first

$$\int_0^1 \int_{\sqrt{y}}^1 x^2 ydxdy = \int_0^1 y\, [\frac{x^3}{3}\Big|_{\sqrt{y}}^1 dy$$

$$= \int_0^1 \left[\frac{y}{3} - \frac{yy^{3/2}}{3}\right] dy = \int_0^1 \left[\frac{y}{3} - \frac{y^{5/2}}{3}\right] dy = [\ \frac{y^2}{6} - \frac{2y^{7/2}}{21}\Big|_0^1$$

$= \frac{1}{6} - \frac{2}{21} = \frac{1}{14}$

Example:-

Evaluate $\int_0^1 \int_0^{x^2} \mathbf{x \cos y \, dydx}$

Solution

1. With y integration first.

$\int_0^1 \int_0^{x^2} x \cos y \, dydx = \int_0^1 \mathrm{x} \; [\sin \mathrm{y}|_0^{x^2} \mathrm{dx} = \int_0^1 \mathrm{x} \; \sin \mathrm{x}^2 \, \mathrm{dx}$

$= [-\frac{1}{2} \cos x^2 \Big|_0^1 = -\frac{1}{2}[\cos 1 - \cos 0] = \frac{1}{2}[1 - \cos 1]$

2. With x integration first.

$\int_0^1 \int_{\sqrt{y}}^1 x \cos y \, dx \, dy = \int_0^1 \left[\cos y [\frac{x^2}{2}\right]_{\sqrt{y}}^1 dy = \int_0^1 \cos y \left(\frac{1}{2} - \frac{y}{2}\right) dy$

$= \int_0^1 \left[\frac{1}{2}\cos y - \frac{y}{2}\cos y\right] dy = \left[\frac{1}{2}\sin \mathrm{y} - \frac{1}{2}(\mathrm{y} \sin \mathrm{y} + \cos \mathrm{y})\right]_0^1$

$\frac{1}{2} \sin 1 - \frac{1}{2}\sin 1 - \frac{1}{2}\cos 1 - \left(\frac{1}{2}\sin 0 - \frac{1}{2}(0 + \cos 0)\right)$

$= -\frac{1}{2}\cos 1 + \frac{1}{2} = \frac{1}{2}(1 - \cos 1)$

Example:-

Evaluate $\int_0^1 \int_y^1 \mathbf{e^{x^2} dxdy}$

Solution

The integration $\int e^{x^2} dx$ cannot be solving analytically, We reverse the order and sketch the region.

$\int_0^1 \int_0^x \mathbf{e^{x^2} dydx} = \int_0^1 \mathbf{e^{x^2}} |_0^x \mathbf{y \, dx} = \int_0^1 \mathbf{e^{x^2} x \; dx}$

$\frac{1}{2} e^{x^2} \Big|_0^1 = \frac{1}{2}(e^1 - e^0) = \frac{1}{2}(e^1 - 1)$

Example:-

Reverse the order of integration and evaluate the resulting integral

1- $\int_0^4 \int_{\sqrt{y}}^2 \mathbf{y \cos x^5 dx \, dy}$

Solution

$\boldsymbol{x = \sqrt{y} \;\; \Rightarrow y = x^2}$

$0 \le x \le 2$ and $0 \le y \le x^2$

$\int_0^4 \int_{\sqrt{y}}^2 \mathrm{y} \cos \mathrm{x}^5 \mathrm{dx} \, \mathrm{dy} = \int_0^2 \int_0^{x^2} y \cos x^5 dydx$

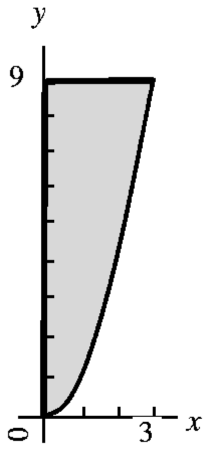
$\int_0^2 \frac{1}{2} y^2 \cos x^5 \,|_0^{x^2} dx = \int_0^2 \frac{1}{2} x^4 \cos x^5 \; dx = \; \frac{1}{2}\; [\frac{\sin x^5}{5}]\Big|_0^2 = \frac{1}{10}\; \sin 32$

2- $\int_0^3 \int_{x^2}^9 x^3 e^{y^3} dydx$

$y = x^2 \;\; \Rightarrow x = \sqrt{y}$

From the figure we get

$0 \le x \le \sqrt{y}$ and $0 \le y \le 9$



$\int_0^3 \int_{x^2}^9 x^3 e^{y^3} dydx = \int_0^9 \int_0^{\sqrt{y}} x^3 e^{y^3} dxdy$

$= \int_0^9 \frac{1}{4} x^4 e^{y^3} \Big|_0^{\sqrt{y}} dy = \int_0^9 \frac{1}{4} y^2 e^{y^3} dy = \frac{1}{4} * \frac{1}{3} [\, e^{y^3} \Big|_0^9$

$= \frac{1}{12} (e^{729} - e^0)$

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c. Central Ideas:

- **Double integral over Rectangular Region**

d. Objectives: after the end of courses the student will be able to:

solve Double integral over Rectangular Region

-

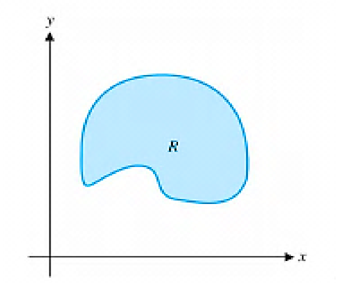
pre test

Q1 Find the area of the (bounded) region between the line $y = 2x$ and the curve $y = \frac{1}{2}x^2$

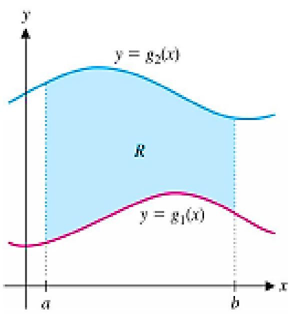
Area Calculated as a Double Integral

Let R be region in the xy – plane , then the area of this region is

Area =A= $\iint\limits_R dA$



1. If R has the shape



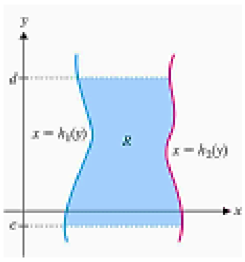
⇒

$$A=\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$$

$$= \int_a^b \left(\begin{matrix}\text{upper}\\ \text{function}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\ \text{function}\end{matrix}\right) dx$$

$$=\int_a^b (g_{2(x)} - g_{1(x)})\,dx$$

2. If R has the shape



⇒

$$A=\int_c^d \int_{h_1(x)}^{h_2(x)} dx\,dy$$

$$=\int_c^d \left(\begin{matrix}\text{right}\\ \text{function}\end{matrix}\right) - \left(\begin{matrix}\text{left}\\ \text{function}\end{matrix}\right) dy$$

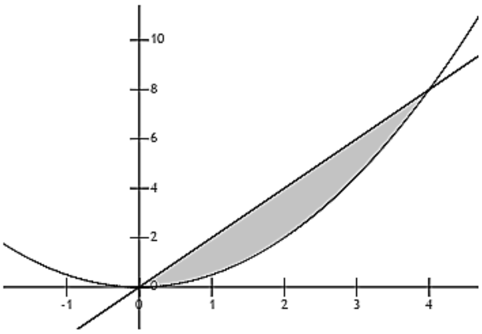
$$=\int_c^d (h_{2(x)} - h_{1(x)})\,dy$$

Example:-

Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$

Solution

$A = \iint_R \, dA = \int_0^{Y} \int_{\frac{x^2}{2}}^{2x} dy\, dx$



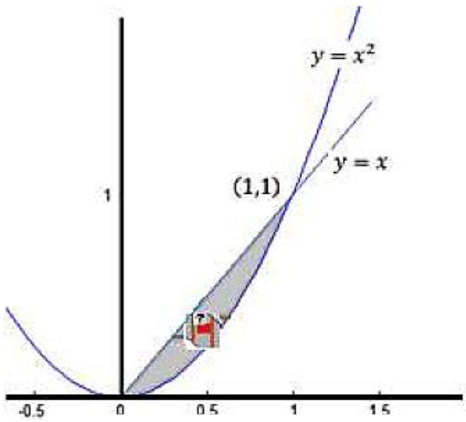
$\int_0^4 (2x - \frac{x^2}{2})dx = (x^2 - \frac{x^3}{6})\Big|_0^4 = \frac{16}{3}$

Example:-

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant

Solution

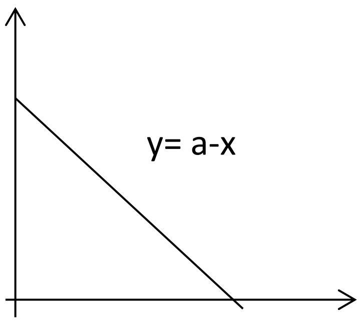
$A = \iint_R \, dA = \int_0^1 \int_{x^2}^{x} dy\, dx$



$\int_0^1 (x - x^2)dx = (\frac{x^2}{2} - \frac{x^3}{3})\Big|_0^1 = \frac{1}{6}$

Example:- **Sketch the region bounded by th** … **nd find it area:**

1- $y = 0, x = 0, x + y = a$



Solution

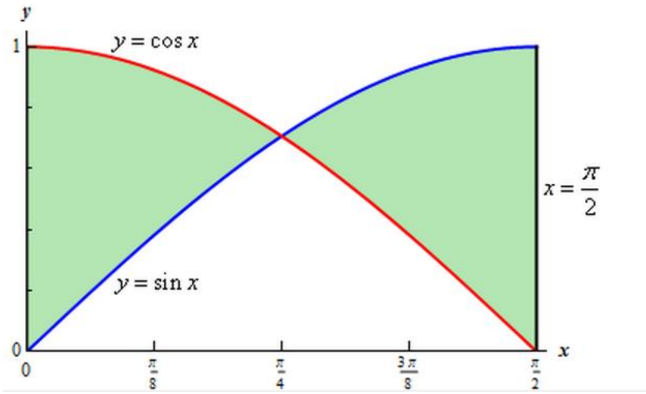
$x + y = a \Rightarrow y = a - x$

$A = \int_0^a \int_0^{a-x} dy\, dx$

$$\int_0^a y\Big|_0^{a-x} dx = \int_0^a (a - x)\ dx = ax - \frac{x^2}{2}\Big|_0^a = a^2 - \frac{a^2}{2}$$

2- $\mathbf{x = \frac{\pi}{2}, y = \cos x, y = \sin x}$

Solution



$$A = \int_0^{\frac{\pi}{4}} \int_{\sin x}^{\cos x} dy\, dx$$

The intersection point will be where

$$\cos x = \sin x$$

The area is then

$$A = \int_0^{\frac{\pi}{4}} \int_{\sin x}^{\cos x} dy\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\sin x}^{\cos x} dy\, dx$$

$$A = \int_0^{\frac{\pi}{4}} \cos x - \sin x\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x\, dx$$

$$A = (\sin x + \cos x)\Big|_0^{\frac{\pi}{4}} + (-\cos x - \sin x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$A = \sqrt{2} - 1 + \sqrt{2} - 1 = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$$

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c. Central Ideas:

Converting Cartesian Integrals to Polar Integrals

d. Objectives: after the end of courses the student will be able to:

- **solve Double integral over Rectangular Region**

pre test

Q1 Evaluate $\int_0^1 \int_0^1 (x^2 + y^2)\, dxdy$

Converting Cartesian Integrals to Polar Integrals

To convert Cartesian integrals to polar integrals, we make the substitution $x = r\cos\theta$ and $y = r\sin\theta$, and replace dy dx with r dr dθ Then we must change the Cartesian limits to polar limits.

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$$

Example:

Evaluate the double integral by polar coordinate $\int\int x^2 + y^2$ dxdy , where R is the region in the first quadrant and bounded by $x^2+y^2=1$.

Solution

$r^2 = x^2+y^2$

$r^2 = 1 \Longrightarrow r = 1$

$$\therefore \int\int x^2 + y^2 \,dxdy = \int_0^{\frac{\pi}{2}}\int_0^1 r^2 r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r^3\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_0^1 d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4}\,d\theta$$

$$= \frac{\theta}{4}\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{8}$$

Example:

Evaluate the double integral by polar coordinate $\int\int e^{x^2+y^2}$ dA , where R is the region in the first quadrant and bounded by $x^2+y^2=1$.

Solution

$$\therefore \int\int e^{x^2+y^2}\,dA = \int_0^{\frac{\pi}{2}}\int_0^1 e^{r^2} r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^1 2re^{r^2}\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} e^{r^2}\Big|_0^1 d\theta \quad = \frac{1}{2}\int_0^{\frac{\pi}{2}}(e^1 - 1)\,d\theta$$

$$= \frac{1}{2}[(e-1)\theta|_0^{\frac{\pi}{2}} = \frac{\pi}{4}(e-1)$$

Example:

Evaluate the integral using polar coordinate $\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2+y^2)dxdy$

Solution

$$\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2+y^2)dxdy = \int_0^{\frac{\pi}{2}} \int_0^2 r^2 r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 r^3 \, dr \, d\theta = \int_0^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_0^2 \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4 \, d\theta = 4\theta\Big|_0^{\frac{\pi}{2}} = 2\pi$$

Example:

Evaluate the double integral by polar coordinate $\int\int 3x+4y^2 \, dA$ Where $x^2+y^2=1$, $x^2+y^2=4$, $y \geq 0$

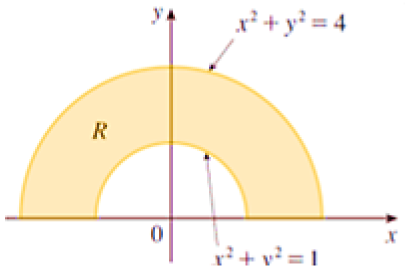
Solution

$\{1 \leq x^2+y^2 \leq 4\}$

x^2+y^2

$1 \Longrightarrow r=1$

$4 \Longrightarrow r=2$



$$\int\int 3x+4y^2 = \int_0^{\pi} \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta) r \, dr \, d\theta$$

$$= \int_0^{\pi} \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta) \, dr \, d\theta$$

$$= \int_0^{\pi} [r^3\cos\theta + r^4\sin^2\theta|_1^2 \, d\theta$$

$$= \int_0^{\pi} [8\cos\theta + 16\sin^2\theta - (\cos\theta + \sin^2\theta)] \, d\theta$$

$$= \int_0^{\pi} [7\cos\theta + 15\sin^2\theta] \, d\theta$$

$$= \int_0^{\pi} [7\cos\theta + \frac{15}{2}(1-\cos 2\theta)] \, d\theta$$

$$= 7\sin\theta + \frac{15}{2}(\theta - \frac{1}{2}\sin 2\theta\Big|_0^{\pi} = \frac{15}{2}\pi$$

H.W

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-

pre test

Q1 Evaluate $\int_0^1 \int_0^1 (x^2 + y^2)\, dxdy$

Surface Area

If $f(x, y, z) = c$ **is surface then :**

S= surface Area = $\iint_R \frac{|\nabla f|}{|\nabla f.\vec{n}|} dA$

Where $\vec{n}$ **is the unit normal vector project on the plane**

R: is the projected region

Example :-**Find the surface area of the upper cut from the sphere** $x^2 + y^2 + z^2 = 2$ **by the cylinder** $x^2 + y^2 = 1$

Sol:-

$$f(x, y, z) = x^2 + y^2 + z^2 = 2$$

$$f_x = 2x \, , f_y = 2y \, , f_z = 2z$$

$$\therefore \overrightarrow{\nabla f} = 2xi + 2yj + 2zk$$

$$\left|\overrightarrow{\nabla f}\right| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{2}$$

Taking $\vec{n} = k$ is a unit vector normal to the x-y plane

$$\overrightarrow{\nabla f}.\vec{n} = 2z \Rightarrow \left|\overrightarrow{\nabla f}.\vec{n}\right| = |2z| = 2z$$

$$\boldsymbol{s = \iint_R \frac{|\nabla \vec{f}|}{|\overrightarrow{\nabla f.n}|} dA = \iint_R \frac{2\sqrt{2}}{2z} dA = \iint_R \frac{\sqrt{2}}{z} dA}$$

$$\boldsymbol{x^2 + y^2 + z^2 = 2 \Rightarrow z^2 = 2 - x^2 - y^2 \Rightarrow z = \sqrt{2 - x^2 - y^2}}$$

$$\boldsymbol{s = \iint_R \frac{\sqrt{2}}{z} dA = \iint_R \frac{\sqrt{2}}{\sqrt{2 - x^2 - y^2}} dA}$$

Since R is the circle $\boldsymbol{x^2 + y^2 = 1}$

$$s = \sqrt{2}\int_0^{2\pi}\int_0^1 \frac{rdr\, d\theta}{\sqrt{2 - r^2}}$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 (2 - r^2)^{-\frac{1}{2}} rdr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \frac{1/2\,(2 - r^2)^{-\frac{1}{2}}}{1/2}\Bigg|_0^1 d\theta = \sqrt{2}\int_0^{2\pi}(1 - \sqrt{2})\, d\theta$$

$$= \sqrt{2}(1 - \sqrt{2})\int_0^{2\pi} d\theta = (2 - \sqrt{2})\theta\Bigg|_0^{2\pi} = (2 - \sqrt{2})2\pi$$

Example :- **Find the surface area cut from the plane $\boldsymbol{2x - y + 3z = 6}$ by the cylinder $\boldsymbol{x^2 + z^2 = 4}$**

Sol:-

$\boldsymbol{f(x, y, z) = 2x - y + 3z} = 6$

$$f_x = 2\,, f_y = -1\,, f_z = 3$$

$$\therefore \overrightarrow{\nabla f} = 2i - j + 3k$$

$$\left|\overrightarrow{\nabla f}\right| = \sqrt{4 + 1 + 9} = \sqrt{14}$$

Taking $\vec{n} = j$is a unit vector normal to the x-zplane

$$\overrightarrow{\nabla f}.\vec{n} = -1 \Rightarrow \left|\overrightarrow{\nabla f}.\vec{n}\right| = |-1| = 1$$

$$\boldsymbol{s = \iint\limits_{R} \frac{|\vec{\nabla} \mathbf{f}|}{\left|\overrightarrow{\nabla \mathbf{f}.\mathbf{n}}\right|} dA = \iint\limits_{R} \sqrt{14}\, dA}$$

Since R is the circle $\boldsymbol{x^2 + z^2 = 4}$

$$s == \int_0^{2\pi}\int_0^2 \sqrt{14} r dr\, d\theta = \sqrt{14}\int_0^{2\pi} \frac{r^2}{2}\Bigg|_0^2 d\theta = \frac{\sqrt{14}}{2}\int_0^{2\pi} r^2 \Bigg|_0^2 d\theta$$

$$= \frac{\sqrt{14}}{2}\int_0^{2\pi} 4\, d\theta = \sqrt{2} = \frac{4\sqrt{14}}{2}\int_0^{2\pi} d\theta$$

$$= 2\sqrt{14}\,\theta \Bigg|_0^{2\pi} = 4\sqrt{14}\,\pi$$

Note

1- $z = f(x,y) = then\ s = \iint_R \sqrt{1 + +(f_x)^2 + (f_y)^2}\ dA$

2- $x = f(y,z) = then\ s = \iint_R \sqrt{1 + +(f_y)^2 + (f_z)^2}\ dA$

3- $y = f(x,y) = then\ s = \iint_R \sqrt{1 + +(f_x)^2 + (f_z)^2}\ dA$

Example :- **Find the surface area of the plane $\boldsymbol{z = x^2 + y^2}$ from $\boldsymbol{z = 1\ to\ z = 4}$**

Sol:-

$\boldsymbol{f(x,y) = x^2 + y^2}$

$$\boldsymbol{f_x = 2x\ , f_y = 2y}$$

$$\boldsymbol{s = \iint\limits_{R} \sqrt{1 + (f_x)^2 + (f_y)^2}\ dA = \iint\limits_{R} \sqrt{1 + 4x^2\ + 4xy^2}\, dA}$$

$$\int_0^{2\pi}\int_1^2 \sqrt{1+4r^{\,2}}\, rdrd\,\theta = \int_0^{2\pi} \frac{1}{8} * \frac{(1+4r^2)^{3/2}}{3/2}\Bigg|_0^2 \, d\theta$$

$$= \frac{1}{12}\int_0^{2\pi} [(1+16)^{3/2} - (5)^{3/2}]d\theta = 4.91\int_0^{2\pi} d\theta = 4.91\,\theta\,\Bigg|_0^{2\pi}$$

$$= 4.91 * 2\pi$$

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand **Multiple Integrals**

c. Central Ideas:

Converting Cartesian Integrals to Polar Integrals

d. Objectives: after the end of courses the student will be able to:

- **solve Double integral over Rectangular Region**

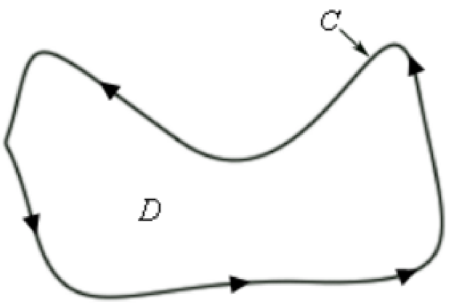
pre test

Q1 Evaluate $\int_0^1 \int_0^1 (x^2 + y^2)\, dxdy$

Greens theorem

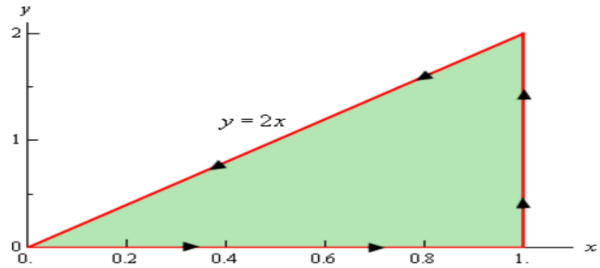
Let C be appositive oriented ,piece wise smooth , simple closed curve and let D be the region enclosed by the curve .If M and N have continuous first partial derivatives on D then

$$\oint_C M dx + N dy = \iint_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} dA$$



Example:-

Use Green's theorem to evaluate $\oint_C xydx + x^2y^3dy$ where C is the triangle with vertices (0,0),(1,0),(1,2) with positive orientation



Solution

$m = \frac{2-0}{1-0} = 2$

$y - y_0 = m(x - x_0)$

$y = 2x$

$M = xy$ and $N = x^2y^3$

$\oint_C xydx + x^2y^3dy = \iint(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dA = \iint(2xy^3 - x)dA$

$\int_0^1 \int_0^{2x} (2xy^3 - x)dydx$

$$\int_0^1 \int_0^{2x} (2xy^3 - x)dydx = \int_0^1 \left(\frac{xy^4}{2} - xy\right)\Big|_0^{2x} dx$$

$$\int_0^1 (8x^5 - 2x^2)dx = \frac{2}{3}$$

Example:-

Evaluate $\oint_C y^3dx - x^3dy$ where C is the positively oriented circle of radius 2 centered at the origin

solution

$M = y^3$ and $N = -x^3$

$$\oint_C y^3dx - x^3dy = \iint(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dA = \iint(-3x^2 - 3y^2)dA$$

$= -3\iint(x^2 + y^2)dA = \int_0^{2\pi}\int_0^2 (r^3)drd\theta$

$$= -3\int_0^{2\pi} \left(\frac{r^4}{4}\right)\Big|_0^2 d\theta = -24\pi$$

H.W:-

1-Evaluate $\oint_C -ydx + x^3dy$ where C are the two circle of radius 2 and radius 1 centered at the origin with positive orientation.

2-Use Green's theorem to evaluate $\oint_C -ydx + xdy$ **where C is the circumference of circle** $x^2 + y^2 = \mathbf{1}$

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand Series

c. Central Ideas:

d. Objectives: after the end of courses the student will be able to:

find the nt term of series

- Test convergie the series

pre test

The following infinite series $\left(\mathbf{12+4+\frac{4}{3}+\frac{4}{9}+\cdots}\right)$ **………… (converges /diverges) because…….**

Sequences and Series

Sequences of Numbers

A ***sequence*** of numbers is a function whose domain is the set of positive inte

Example

$0, \quad 1, \quad 2, \quad . \quad . \quad . \quad n-1, \quad . \quad . \quad .$ for a sequence whose defining rule is $a_n =$

$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad . \quad . \quad . \quad \frac{1}{n}, \quad . \quad . \quad .$ for a sequence whose defining rule is $a_n =$

The index n is the ***domain*** of the sequence. While the numbers in the ***rang*** sequence are called the ***terms*** of the sequence, and the number a_n being called ***term***, or ***the term with index*** n.

Example $\quad a_n = \frac{n+1}{n}$ then the terms are

$$\begin{array}{ccccc} 1^{st}\ term & 2^{nd}\ term & 3^{rd}\ term & & n^{th}\ term \\ a_1 = 2, & a_2 = \frac{3}{2}, & a_3 = \frac{4}{3}, & . \quad . \quad . & a_n = \frac{n+1}{n}, \quad . \quad . \quad . \end{array}$$

and we use the notation $\{a_n\}$ as the sequence a_n.

Example

Find the first five terms of the following:

(a) $\left\{\frac{2n-1}{3n+2}\right\}$, (b) $\left\{\frac{1-(-1)^n}{n^3}\right\}$, (c) $\left\{(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}\right\}$

Solution

(a) $\frac{1}{5}, \quad \frac{3}{8}, \quad \frac{5}{11}, \quad \frac{7}{14}, \quad \frac{9}{17}$ **(b)** $2, \quad 0, \quad \frac{2}{27}, \quad 0, \quad \frac{2}{125}$

(c) $x, \quad \frac{-x^3}{3!}, \quad \frac{x^5}{5!}, \quad \frac{-x^7}{7!}, \quad \frac{x^9}{9!}$

Example

Find the nth-term of the following:

(a) $0, \ \frac{1}{2}, \ \frac{2}{3}, \ \frac{3}{4},$ (b) $0, \ \frac{\ln 2}{2}, \ \frac{\ln 3}{3}, \ \frac{\ln 4}{4},$ (c) $0, \ \frac{1}{4}, \ \frac{2}{9}, \ \frac{3}{16},$

(d) $2, \ 1, \ \frac{2^3}{3^2}, \ \frac{2^4}{4^2}, \ \frac{2^5}{5^2}$

Solution

(a) $a_n = \frac{n-1}{n},$ **(b)** $a_n = \frac{\ln n}{n},$ **(c)** $a_n = \frac{n-1}{n^2},$ **(d)** $a_n = \frac{2^n}{n^2}$

Convergence of Sequences

The fact that $\{a_n\}$ converges to L is written as

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \ \text{ as } \ n \to \infty$$

and we call the limit of the sequence $\{a_n\}$. If no such limit exists, we say that $\{a_n\}$ diverges.

From that we can say that

1) $\lim_{n\to\infty} a_n = L$ (Conv.)

2) $\lim_{n\to\infty} a_n = \infty$ (Div.)

3) $\lim_{n\to\infty} a_n = \begin{cases} L_1 \\ L_2 \end{cases}$ (Div.)

Also, if $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$ both exist and are finite, then

i) $\lim_{n\to\infty}\{a_n + b_n\} = A + B$

ii) $\lim_{n\to\infty}\{ka_n\} = kA$

iii) $\lim_{n\to\infty}\{a_n \cdot b_n\} = A \cdot B$

iv) $\lim_{n\to\infty}\left\{\frac{a_n}{b_n}\right\} = \frac{A}{B}$, provided $B \neq 0$ and b_n is never 0

Example

Test the convergence of the following:

(a) $\left\{\frac{1}{n}\right\}$, (b) $\{1+(-1)^n\}$, (c) $\{n^2\}$, (d) $\{\sqrt{n+1}-\sqrt{n}\}$,

(e) $\left\{\frac{3n^2-5n}{5n^2+2n+6}\right\}$, (f) $\left\{\frac{3n^2-4n}{2n-1}\right\}$, (g) $\left\{\left(\frac{2n-3}{3n-7}\right)^4\right\}$, (h) $\left\{\frac{2n^5-4n^2}{3n^7+n^2-10}\right\}$,

(i) $\left\{\frac{2^n}{5n}\right\}$, (j) $\left\{\frac{\ln n}{e^n}\right\}$

Solution

(a) $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$ ***(Conv.)***

(b) $\lim_{n\to\infty}\left(1+(-1)^n\right) = 1+\lim_{n\to\infty}(-1)^n = \begin{cases} 0 & n \ \ odd \\ 2 & n \ \ even \end{cases}$ ***(Div.)***

(c) $\lim_{n\to\infty}\left(n^2\right) = \infty$ ***(Div.)***

(d) $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = \lim_{n\to\infty}\left(\left(\sqrt{n+1}-\sqrt{n}\right)\times\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right) = \lim_{n\to\infty}\left(\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right)$

$$= \lim_{n\to\infty}\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) = \frac{1}{\infty+\infty} = 0$$ ***(Conv.)***

Infinite Series

Infinite series are sequences of a special kind: those in which the n^{th}-term is th sum of the first n terms of a related sequence.

Example

Suppose that we start with the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$$

If we denote the above sequence as a_n, and the resultant sequence of the series as s_n then

$$s_1 = a_1 = 1,$$

$$s_2 = a_1 + a_2 = 1 + \frac{1}{2} = \frac{3}{2},$$

$$s_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4},$$

as the first three terms of the sequence $\{s_n\}$.

When the sequence $\{s_n\}$ is formed in this way from a given sequence $\{a_n\}$ by th rule

$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

the result is called an ***Infinite Series.***

- The number $s_n = \sum_{k=1}^{n} a_k$ is called the n^{th} *partial sum* of the series.
- Instead of $\{s_n\}$, we usually write $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$.
- The series $\sum a_n$ is said to *converge* to a number L if and only if

$$L = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

in which case we call L the sum of the series and write

$$\sum_{n=1}^{\infty} a_n = L \qquad \text{or} \qquad a_1 + a_2 + \ldots + a_n + \ldots = L$$

If no such limit exists, the series is said to ***diverge***.

Geometric Series

A series of the form

$$a + ar + ar^2 + ar^3 + \ldots + +ar^{n-1} + \ldots$$

is called a ***Geometric Series***. The ratio of any term to the one before it is r. If $|r| < 1$, the geometric series converges to $a/(1-r)$. If $|r| \geq 1$, the series diverges unless $a = 0$. If $a = 0$, the series converges to 0.

Example

Geometric series with $a = \frac{1}{9}$ and $r = \frac{1}{3}$.

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots = \frac{1}{9}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\right) = \frac{1/9}{1-(1/3)} = \frac{1}{6}$$

Geometric series with $a = 4$ and $r = -\frac{1}{2}$.

$$4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \ldots = 4\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \ldots\right)$$

$$= \frac{4}{1+(1/2)} = \frac{8}{3}$$

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand Series

c. Central Ideas:

Test Convergence of series with Non- negative trems

d. Objectives: after the end of courses the student will be able to:

Test Convergence of series with Non- negative trems
pre test

The infinite series $\sum_{n=1}^{\infty}\frac{n+1}{n}$ ....(converges /diverges) because..........[using the n^{th} – Term test]

Test Convergence of Series with Non-negative Terms

1) The n^{th}- Term Test

- If $\lim_{n\to\infty} a_n \neq 0$, or if $\lim_{n\to\infty} a_n$ fails to exist, then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.
- If $\lim_{n\to\infty} a_n = 0$, then the test fails.

From the above, it can not be concluded that if $a_n \to 0$ then $\sum_{n=1}^{\infty} a_n$ converges. The series $\sum_{n=1}^{\infty} a_n$ may diverge even though $a_n \to 0$. Thus $\lim_{n\to\infty} a_n = 0$ is a necessary but not a sufficient condition for the series $\sum_{n=1}^{\infty} a_n$ to converge.

Examples

$\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$,

$\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1 \neq 0$,

$\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} (-1)^{n+1}$ does not exist,

$\sum_{n=1}^{\infty} \frac{n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{n}{2n+5} = \frac{1}{2} \neq 0$,

$\sum_{n=1}^{\infty} \frac{1}{n}$ can not be tested by the n$^{\text{th}}$-term test for divergence because $\frac{1}{n} \to 0$.

2) The Integral Test

Let the function $y = f(x)$, obtained by introducing the continuous variable x in place of the discrete variable n in the n$^{\text{th}}$-term of the positive series $\sum_{n=1}^{\infty} a_n$, then

$$\int_1^{\infty} f(x)dx = \begin{cases} +\infty & Div. \\ -\infty & Div. \\ -\infty < c < \infty & Conv. \end{cases}$$

Example

Test the convergence of

(a) $\sum_{n=1}^{\infty}\frac{1}{e^n}$, (b) $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

Solution

(a) $\int_1^{\infty} e^{-x}dx = -e^{-x}\Big|_1^{\infty} = -(e^{-\infty} - e^{-1}) = \frac{1}{e}$ ***(Conv.)***

(b) $\int_2^{\infty}\frac{1}{x(\ln x)^2}dx = \int_2^{\infty}\frac{1/x}{(\ln x)^2}dx = \frac{-1}{\ln x}\Big|_2^{\infty} = \frac{-1}{\infty} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$ ***(Conv.)***

1. Overview

a. Target Population: For students of second stage in college electrical engineering technical college in middle technical university.

b. Rationale: we will understand Series

c. Central Ideas:

d. Objectives: after the end of courses the student will be able to:

- **test the convergies of series**

pre test
Q1
Use the integral test to determine the convergence or divergence of the series

3) The Ratio Test

Let $\sum a_n$ be a series with positive terms, and suppose that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\rho$$

Then

- The series converges if $\rho<1$,
- The series diverges if $\rho>1$,
- The series may converge or it may diverge if $\rho=1$. (Test fails)

The Ratio Test is often effective when the terms of the series contain factorials of expressions involving n or expressions raised to a power involving n.

Example

Test the following series for convergence or divergence , using the Ratio Test.

(a) $\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$, (b) $\sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$, (c) $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$, (d) $\sum_{n=1}^{\infty} \frac{n!}{3^n}$, (e) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Solution

(a) If $a_n = \frac{n!n!}{(2n)!}$, then $a_{n+1} = \frac{(n+1)!(n+1)!}{(2n+2)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)(2n+1)(2n)!} = \frac{(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \frac{n+1}{4n+2} \to \frac{1}{4} < 1 \qquad \textit{(Conv.)}$$

(b) If $a_n = \frac{4^n n!n!}{(2n)!}$, then $a_{n+1} = \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)(2n+1)(2n)!} \times \frac{(2n)!}{4^n n!n!} = \frac{4(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \frac{2(n+1)}{2n+1} \to 1 \qquad \textit{(Test fails)}$$

(c) If $a_n = \frac{2^n+5}{3^n}$, then $a_{n+1} = \frac{2^{n+1}+5}{3^{n+1}}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1}+5)/3^{n+1}}{(2^n+5)/3^n} = \frac{1}{3} \times \frac{2^{n+1}+5}{2^n+5}$$

$$= \frac{1}{3} \times \left(\frac{2+5\times 2^{-n}}{1+5\times 2^{-n}} \right) \to \frac{1}{3} \times \frac{2}{1} = \frac{2}{3} < 1 \qquad \textit{(Conv.)}$$

(d) If $a_n = \dfrac{n!}{3^n}$, then $a_{n+1} = \dfrac{(n+1)!}{3^{n+1}}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{3^{n+1}} \times \frac{3^n}{n!} = \frac{n+1}{3} \to \infty > 1 \qquad \textbf{\textit{(Div.)}}$$

(e) If $a_n = \dfrac{n^n}{n!}$, then $a_{n+1} = \dfrac{(n+1)^{n+1}}{(n+1)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n} = \frac{(n+1)^n (n+1) n!}{(n+1) n! n^n}$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e^1 = 2.7 > 1 \qquad \textbf{\textit{(Div.)}}$$

4) The n^{th} Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n > n_0$ and suppose that

$$\sqrt[n]{a_n} \to \rho$$

Then

- The series converges if $\rho < 1$.
- The series diverges if $\rho > 1$.
- The test is not conclusive if $\rho = 1$.

Example

Test the convergence of the following series using the n^{th} Root Test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$, (c) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$, (d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$, (e) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

Solution

(a) $\sqrt[n]{\dfrac{1}{n^n}} = \dfrac{1}{n} \to 0 < 1$ ***(Conv.)***

(b) $\sqrt[n]{\dfrac{2^n}{n^2}} = \dfrac{2}{\sqrt[n]{n^2}} = \dfrac{2}{\left(\sqrt[n]{n}\right)^2} \to \dfrac{2}{1^2} = 2 > 1$ ***(Div.)***

(c) $\sqrt[n]{\left(1 - \dfrac{1}{n}\right)^n} = \left(1 - \dfrac{1}{n}\right) \to 1$ ***(Test fails)***

(d) $\sqrt[n]{\left(\dfrac{n}{n+1}\right)^{n^2}} = \left(\dfrac{n}{n+1}\right)^{\frac{n^2}{n}} = \left(\dfrac{n}{n+1}\right)^n = \left(\dfrac{1}{1+1/n}\right)^n \to \dfrac{1}{e} = \dfrac{1}{2.7} < 1$ ***(Conv.)***

(e) $\sqrt[n]{\left(\dfrac{2n}{n+1}\right)^n} = \dfrac{2n}{n+1} \to 2 > 1$ ***(Div.)***