Root Locus Technique

The root locus is a graphical procedure for determining the poles of a closed-loop system given the poles and zeros of a forward-loop system. Graphically, the locus is the set of paths in the complex plane traced by the closed-loop poles as the root locus gain is varied from zero to infinity. In mathematical terms, given a forward-loop transfer function,

$$K \, G(s)$$

where K is the root locus gain, and the corresponding closed-loop transfer function

$$\frac{K \, G(s)}{1 + K \, G(s)}$$

The root locus is the set of paths traced by the roots of $1 + K \, G(s) = 0$ as K varies from zero to infinity. As K changes, the solution to this equation changes. This equation is called the *characteristic equation*. The roots to the equation are the poles of the forward-loop transfer function. The equation defines where the poles will be located for any value of the *root locus gain*, K. In other words, it defines the characteristics of the system behavior for various values of controller gain.
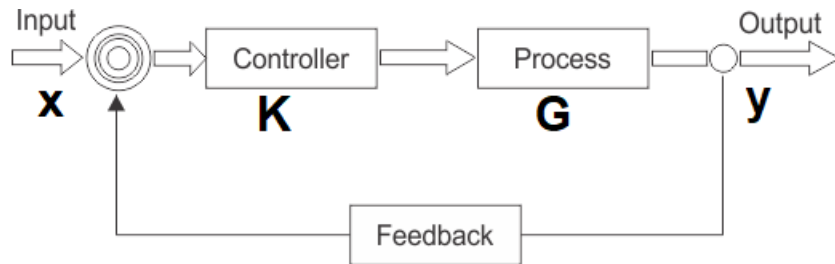

Figure (1): Block diagram for closed loop control system

Characteristic Equation

The *characteristic equation* of a system is based upon the transfer function that models the system. It contains information needed to determine the response of a dynamic system. There is only one characteristic equation for a given system.

Root locus gain

The root locus gain, typically denoted as K, is a gain of the forward-loop system. While determining the root locus, this gain is varied from 0 to infinity. Note that the corresponding variations in the poles of the closed-loop system determine the root locus. As the gain moves from 0 to infinity, the poles move from the forward-loop poles along the locus toward forward-loop zeros or infinity. In block diagram form (see Figure (1)), the root locus gain is located in the forward loop, before the system.

Angle Criterion

The *angle criterion* is used to determine the departure angles for the parts of the root locus near the open-loop poles and the arrival angles for the parts of the root locus near the open-loop zeros. When used with the *magnitude criterion*, the angle criterion can also be used to determine whether or not a point in the s-plane is on the root locus. The angle criterion is defined as on the root locus,

$$\angle K\, G(s) = -180^o$$

Note that +180° could be used rather than -180°. The use of -180° is just a convention. Since +180° and -180° are the same angle, either produces the same result. The angle criterion is a direct result of the definition of the root locus; it is another way to express the locus requirements. The root locus is defined as the set of roots that satisfy the characteristic equation $1 + K\, G(s) = 0$, or, equivalently, $K\, G(s) = -1$, taking the phase of each side of the equation yields the angle criterion.

Magnitude Criterion

The *magnitude criterion* is used to determine the locations of a set of roots in the s-plane for a given value of K. Mathematically, the magnitude criterion is $|K\,G(s)| = 1$. The magnitude criterion is a direct result of the definition of the root locus; it is another way to express the locus requirements. The root locus is defined as the set of roots that satisfy the characteristic equation $1 + K\,G(s) = 0$, or, equivalently, $K\,G(s) = -1$, taking the magnitude of each side of the equation yields the magnitude criterion.

Angle of departure

The *angle of departure* is the angle at which the locus leaves a pole in the s-plane. The *angle of arrival* is the angle at which the locus arrives at a zero in the s-plane. By convention, both types of angles are measured relative to a ray starting at the origin and extending to the right along the real axis in the s-plane. Both arrival and departure angles are found using the *angle criterion*.

Break Point

Break points occur on the locus where two or more loci converge or diverge. Break points often occur on the real axis, but they may appear anywhere in the s-plane. The loci that approach/diverge from a break point do so at angles spaced equally about the break point. The angles at which they arrive/leave are a function of the number of loci that approach/diverge from the break point.

Closed-Loop

A closed-loop system includes feedback. The output from the system is fed back through a controller into the input to the system. If **G**u(s) is the transfer function of the uncontrolled system, and **G**c(s) is the transfer function of the controller, and a unity feedback is used, then the closed loop system can be represented in block diagram form as

$$Y(s) = \frac{G_c(s)\, G_u(s)}{1 + G_c(s)\, G_u(s)}\, X(s)$$
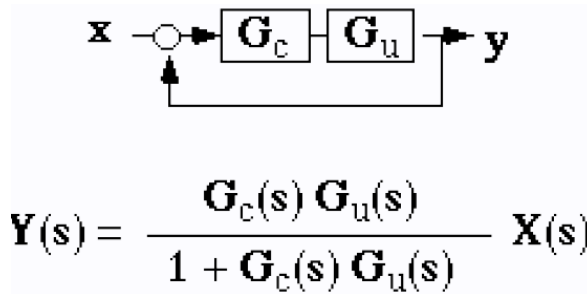
Sometimes a transfer function, **H**(s), is included in the feedback loop. In block diagram form, this can be represented as

$$Y(s) = \frac{G(s)}{1 + H(s)G(s)}\, X(s)$$
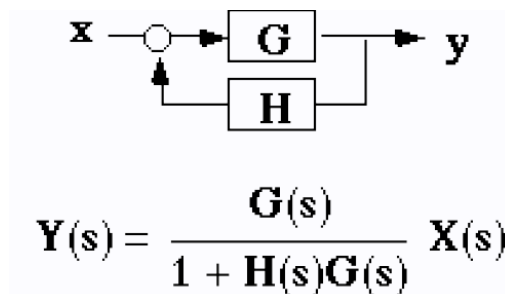
Complex-Plane (s-plane)

The *s-plane* or *complex plane* is a two-dimensional space defined by two orthogonal axes, the real number axis and the imaginary number axis. A point in the s-plane represents a complex number. When talking about control systems, complex numbers are typically represented by the letter s (thus the 's'-plane). Each complex number s has both a real component, typically represented by sigma, and an imaginary component, typically represented by omega.
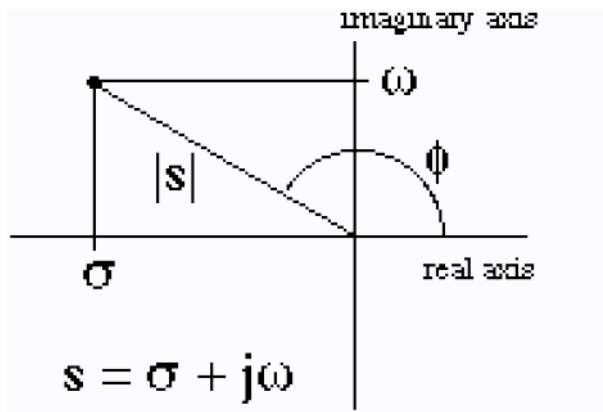
$$s = \sigma + j\omega,$$

Any point in the complex plane has an angle (or phase) and magnitude defined as

$$\angle s = \tan^{-1}\frac{\omega}{\sigma} \qquad |s| = \sqrt{\sigma^2 + \omega^2}$$

phase   magnitude

Graphically, each complex number s is plotted in the s-plane as follows

$$s = \sigma + j\omega$$

Forward Loop

A forward-loop system is a part of a controlled system. As the name suggests, it is the system in the "forward" part of the block diagram. Typically, a forward-loop includes the uncontrolled system cascaded with the controller. For a system with controller $\mathbf{G}c(s)$ and system $\mathbf{G}u(s)$, the block diagram and transfer function of the forward-loop are

$$Y(s) = \mathbf{G}_c(s)\,\mathbf{G}_u(s)\,X(s)$$
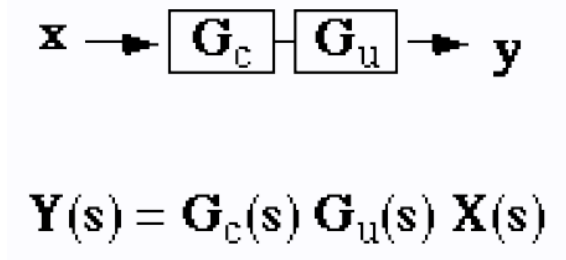
Note that closing a loop around this controller and system using a unity feedback gain yields the closed-loop system.

$$Y(s) = \frac{\mathbf{G}_c(s)\,\mathbf{G}_u(s)}{1 + \mathbf{G}_c(s)\,\mathbf{G}_u(s)}\,X(s)$$
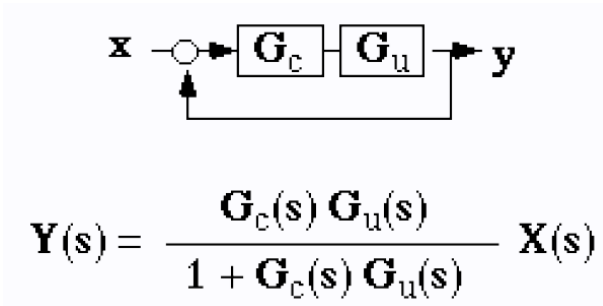
Constructing the Locus

This section outlines the steps to creating a root locus and illustrates the important properties of each step in the process. By the end of this section you should be able to sketch a root locus given the forward-loop poles and zeros of a system. Using these steps, the locus will be detailed enough to evaluate the stability and robustness properties of the closed-loop controller.

Step 1: Open-Loop Roots

Start with the forward-loop poles and zeros. Since the locus represents the path of the roots (specifically, paths of the closed-loop poles) as the root locus gain is varied, we start with the forward-loop configuration, i.e. the location of the roots when the gain of the closed-loop system is 0. Each locus starts at a forward-loop pole and ends at a forward-loop zero. If the system has more poles than zeros, then some of the loci end at zeros located infinitely far from the poles.

Step 2: Real Axis Crossings

Many root loci have paths on the real axis. The real axis portion of the locus is determined by applying the following rule:

If an odd number of forward-loop poles and forward-loop zeros lie to the right of a point on the real axis, that point belongs to the root locus.

Note that the real axis section of the root locus is determined entirely by the number of forward-loop poles and zeros and their relative locations. Since the final root locus is always symmetric about the real axis (think about it), the real axis part is pretty easy to do.

Step 3: Asymptotes

The asymptotes indicate where the poles will go as the gain approaches infinity. For systems with more poles than zeros, the number of asymptotes is equal to the number of poles minus the number of zeros. In some systems, there are no asymptotes; when the number of poles is equal to the number of zeros then each locus is terminated at a zero

rather than asymptotically to infinity. The asymptotes are symmetric about the real axis, and they stem from a point defined by the relative magnitudes of the open-loop roots. This point is called the centroid. Note that it is possible to draw a root locus for systems with more zeros than poles, but such systems do not represent physical systems. In these cases, you can think of some of the poles being located at infinity.

Step 4: Breakpoints

Break points occur where two or more loci join then diverge. Although they are most commonly encountered on the real axis, they may also occur elsewhere in the complex plane. Each break point is a point where a double (or higher order) root exists for some value of K. Mathematically, given the root locus equation

$$1 + K\,G(s) = 0$$

where the transfer function **G**(s) consists of a numerator, **A**(s), and denominator, **B**(s), then the break points can be determined from the roots of

$$\frac{dK}{ds} = \frac{B(s)A'(s) - B'(s)A(s)}{A^2(s)} = 0$$

If K is real and positive at a value s that satisfies this equation, then the point is a break point. There will always be an even number of loci around any break point; for each locus that enters the locus, there must be one that leaves.

Step 5: Angles of Departure/Arrival

The angle criterion determines which direction the roots move as the gain moves from zero (angles of departure, at the forward-loop poles) to infinity (angles of arrival, at the forward-loop zeros). An angle of departure/arrival is calculated at each of the complex forward-loop poles and zeros.

Step 6: Axis Crossings

The points where the root locus intersects the imaginary axis indicate the values of K at which the closed loop system is marginally stable. The closed loop system will be unstable for any gain for which the locus is in the right half plane of the complex plane. If the root locus crosses the imaginary axis from left to right at a point where K=K0 and then stays completely in the right half-plane, then the closed-loop system is unstable for all K>K0. Therefore, knowing the value of K0 is very useful. Some systems are particularly nasty when their locus dips back and forth across the imaginary axis. In these systems, increasing the root locus gain will cause the system to go unstable initially and then becomes stable again.

Step 7: Sketch the Locus

The complete root locus can be drawn by starting from the forward-loop poles, connecting the real axis section, break points, and axis crossings, then ending at either the
forward-loop zeros or along the asymptotes to infinity and beyond. If your hand-drawn locus is not detailed enough to determine the behavior of your system, then you may want to use Matlab or some other computer tool to calculate the locus exactly.

Calculating the Gain

The root locus shows you graphically how the system roots will move as you change the root locus gain. Often, however, one must determine the gain at critical points on the locus, such as points where the locus crosses the imaginary axis. The magnitude criterion is used to determine the value of the root locus gain, K, at any point on the root locus. The gain is calculated by multiplying the lengths of the distance between each pole to the point then dividing that by the product of the lengths of the distance between each zero and the point.

$$K = \frac{\text{product of lengths between points } s \text{ to poles}}{\text{product of lengths between point } s \text{ to zeros}}$$

Consider the system with transfer function

$$\frac{1}{s^2 + 2s - 3}$$

and corresponding root locus
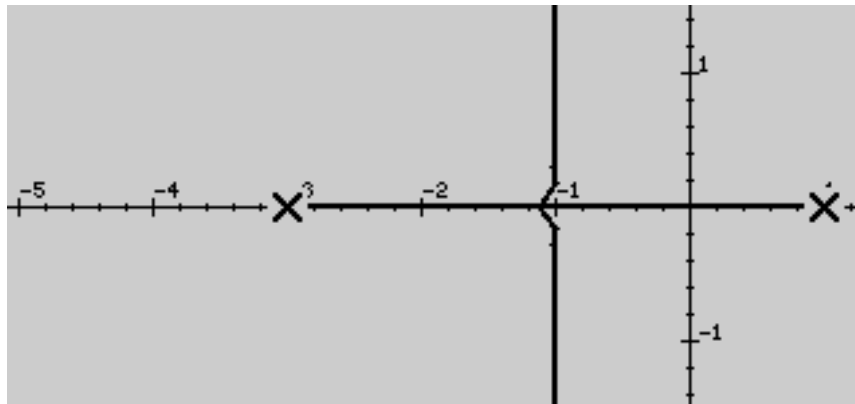
The gain at the point (-1, 0) is thus

$$K = \frac{(2)(2)}{1}$$

The root locus gain for other points on the locus are given in the following table:

Coordinate	gain
0.75, 0	0.9375
0.5, 0	1.75
0.25, 0	2.4375
0, 0	4
0, 0.25	4.0625
0, 0.5	4.25
0, 0.75	4.5625
0, 1	5
0, 2	8
0, 3	13

Note that a linear change in position on the locus usually does not correspond to a linear change in the root locus gain.

Example: Sketch the root loci for the system shown in Figure below, the gain K is assumed to be positive. Observe that for small or large value of K the system is overdamped and for medium values of K it is underdamped.

Solution: The procedure of plotting the root loci is as follows:

1- Locate the poles and zeros on the complex plane. Root loci exist on the negative real axis between 0 and -1 and between -2 and -3.

2- The number of open-loop poles and that of finite zeros are the same. This means that there are no asymptotes in the complex region of s-plane.

3- Determine the breakaway and break-in points. The characteristic equation of the system is

$$1 + \frac{K(s+2)(s+3)}{s(s+1)} = 0 \text{ or } K = -\frac{s(s+1)}{(s+2)(s+3)}$$

The breakaway and break-in points are determined from

$$\frac{dK}{ds} = -\frac{(2s+1)(s+2) - s(s+1)(2s+5)}{[(s+2)(s+3)]^2}$$

$$= -\frac{4(s+0.634)(s+2.366)}{[(s+2)(s+3)]^2} = 0$$

As follows: $s = -0.634,\quad s = -2.366,$

Notice that both points are on root loci. Therefore, they are actual breakaway or break-in points. At point $s = -0.634$, the value of K is

$$K = \frac{(-0.634)(0.366)}{(1.366)(2.366)} = 0.0718$$

Similarly, at $s = -2.366$,

$$K = \frac{(-2.366)(-1.366)}{(-0.366)(0.634)} = 14$$

(Because point $s = -0.634$ lies between two poles, it is a breakaway point, and because point $s = -2.366$ lies between two zeros, it is break-in point.

4- Determine the sufficient number of points that satisfy the angle condition. (It can be found that the root loci involve a circle with center at -1.5 that passes through the breakaway and break-in points). The root-locus plot is as shown in figure below.

Note that this system is stable for my positive value of K since all the root loci lie in the left half s-plane. Small values of K $(0 < K < 0.0718)$ correspond to an overdamped system. Medium values of K $(0.0718 < K < 14)$ correspond to an underdamped system. Finally, large values K $(14 < K)$ correspond to an overdamped system. With a large value of K, the steady state can be reached in much shorter time than with a small value of K. The value of K should be adjusted so that system performance is optimum according to a given performance index.
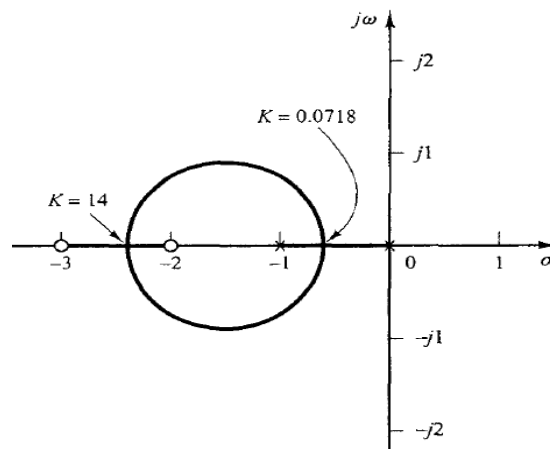
Root Locus Construction Steps

1. Start at the transfer function of the forward loop (T.F = G(s) H(s)), find its poles, zeros, the number of poles (P), and the number of zeros (Z).

2. Find the real axis crossings according to the rule "a point on the real axis lies on the locus if the number of open-loop (poles + zeros) on the real axis to the right half of s-plane is odd.

3. Calculate the number of asymptotes (No. of asymptotes = P – Z).

4. Calculate the angle of asymptotes

$$\text{If P > Z then} \quad \theta_A = \frac{(2q+1)180^0}{P-Z}, \quad q = 0, 1, ..., P-Z-1$$

$$\text{If Z > P then} \quad \theta_A = \frac{(2q+1)180^0}{Z-P}, \quad q = 0, 1, ..., Z-P-1$$

If P = Z then there is no asymptotes

5. Calculate the centroid where the asymptotes meet the real axis:

$$\sigma_c = \frac{\sum real\ P - \sum real\ Z}{P-Z}$$

6. Calculate the Breakaway and Break in points by solving $\frac{dK}{ds} = 0$.

7. Find the intersection points with the imaginary axis using Routh's array.

8. If there is any pole or zero has an imaginary part, then calculate the angle of departure from an open loop pole or the angle of arrival at an open loop zero.
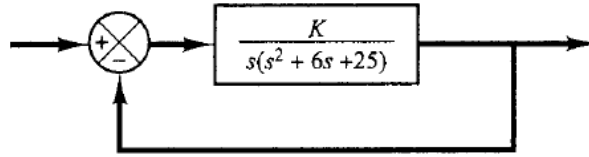
$\varphi_P =$

$180^0 -$

$[sum\ of\ the\ other\ GH\ pole\ angles\ to\ the\ pole\ under\ consideration] +$

$[sum\ of\ the\ GH\ zero\ angles\ to\ the\ pole]$

Example: Sketch the root loci of the control system shown in Figure below.

Solution: The open-loop poles are located at s = 0, s = -3 + j4, and s = -3 - j4. A root locus branch exists on the real axis between the origin and $-\infty$. There are three asymptotes for the root loci. The angles of asymptotes are

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{3} = 60°, -60°, 180°$$

$$s = -\frac{0 + 3 + 3}{3} = -2$$

Next we check the breakaway and break-in points. For this system we have

$$K = -s(s^2 + 6s + 25)$$

Now we set

$$\frac{dK}{ds} = -(3s^2 + 12s + 25) = 0$$

which yields

$$s = -2 + j2.0817, \qquad s = -2 - j2.0817$$

Notice that at points $s = -2 \pm j2.0817$ the angle condition is not satisfied. Hence, they are neither breakaway nor break-in points. In fact, if we calculate the value of K, we obtain

$$K = -s(s^2 + 6s + 25)\bigg|_{s=-2\pm j2.0817} = 34 \pm j18.04$$

(To be an actual breakaway or break-in point, the corresponding value of K must be real and positive.)

The angle of departure from the complex pole in the upper half s plane is

$$\theta = 180° - 126.87° - 90°$$

or

$$\theta = -36.87°$$

The points where root-locus branches cross the imaginary axis may be found by substituting $s = j\omega$ into the characteristic equation and solving the equation for ω and K as follows: Noting that the characteristic equation is

$$s^3 + 6s^2 + 25s + K = 0$$

we have

$$(j\omega)^3 + 6(j\omega)^2 + 25(j\omega) + K = (-6\omega^2 + K) + j\omega(25 - \omega^2) = 0$$

which yields

$$\omega = \pm 5, \qquad K = 150 \qquad \text{or} \qquad \omega = 0, \qquad K = 0$$

Root-locus branches cross the imaginary axis at $\omega = 5$ and $\omega = -5$. The value of gain K at the crossing points is 150. Also, the root-locus branch on the real axis touches the imaginary axis at $\omega = 0$. Figure 6–40(b) shows a root-locus plot for the system.

It is noted that if the order of the numerator of $G(s)H(s)$ is lower than that of the denominator by two or more, and if some of the closed-loop poles move on the root locus toward the right as gain K is increased, then other closed-loop poles must move toward the left as gain K is increased. This fact can be seen clearly in this problem. If the gain K is increased from $K = 34$ to $K = 68$, the complex-conjugate closed-loop poles are moved from $s = -2 + j3.65$ to $s = -1 + j4$; the third pole is moved from $s = -2$ (which corresponds to $K = 34$) to $s = -4$ (which corresponds to $K = 68$). Thus, the movements of two complex-conjugate closed-loop poles to the right by one unit cause the remaining closed-loop pole (real pole in this case) to move to the left by two units.