No.4

5 KARNAUGH MAP MINIMIZATION (K-map)

A Karnaugh map provides a systematic method for simplifying Boolean expressions and, if properly used, will produce the simplest SOP or POS expression possible, known as the minimum expression. As you have seen, the effectiveness of algebraic simplification depends on your familiarity with all the laws, rules, and theorems of Boolean algebra and on your ability to apply them. The Karnaugh map, on the other hand, provides a "cookbook" method for simplification.

A Karnaugh map is similar to a truth table because it presents all of the possible values of input variables and the resulting output for each value. Instead of being organized into columns and rows like a truth table, the Karnaugh map is an array of cells in which each cell represents a binary value of the input variables. The cells are arranged in a way so that simplification of a given expression is simply a matter of properly grouping the cells. Karnaugh maps can be used for expressions with two, three, four. and five variables. Another method, called the Quine-McClusky method can be used for higher numbers of variables.

The number of cells in a Karnaugh map is equal to the total number of possible input variable combinations as is the number of rows in a truth table. For three variables, the number of cells is $2^3 = 8$. For four variables, the number of cells is $2^4 = 16$.

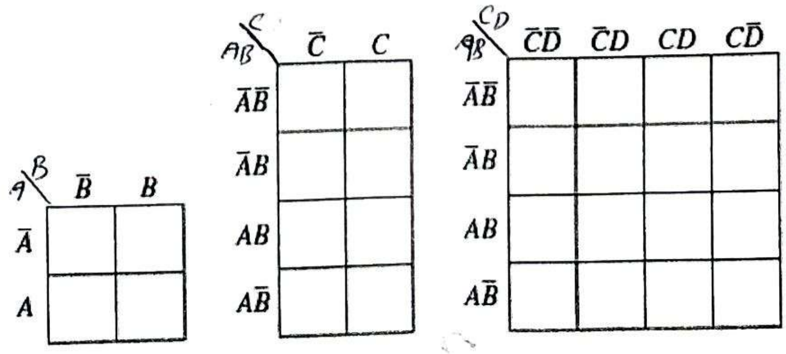
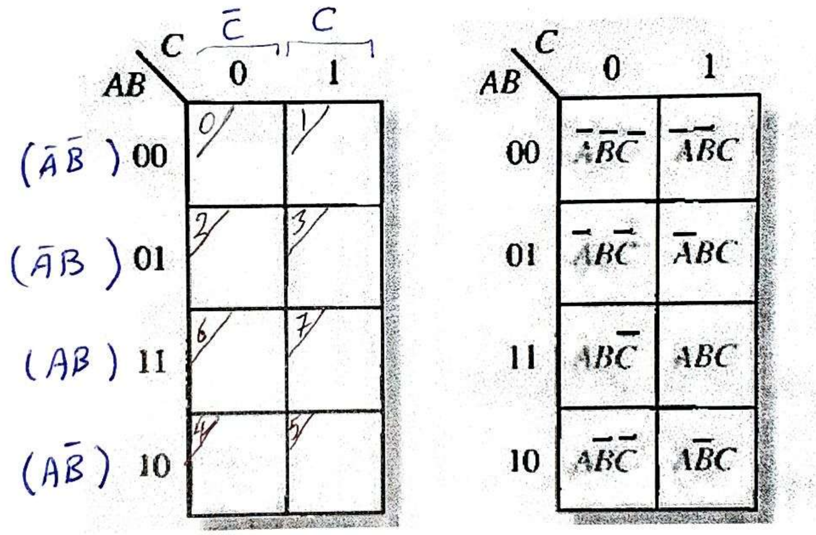


Figure 1 Two-, three-, and four-variable Karnaugh maps.

The 3-Variable Karnaugh Map

The 3-variable Karnaugh map is an array of eight cells. as shown in Fig.(5-1)(a). In this case, A, B, and C are used for the variables although other letters could be used. Binary values of A and B are along the left side (notice the sequence) and the values of C are across the top. The value of a given cell is the binary values of A and B at the left in the same row combined with the value of C at the top in the same column. For example, the cell in the upper left corner has a binary value of 000 and the cell in the lower right corner has a binary value of 101. Fig.(5-1)(b) shows the standard product terms that are represented by each cell in the Karnaugh map.

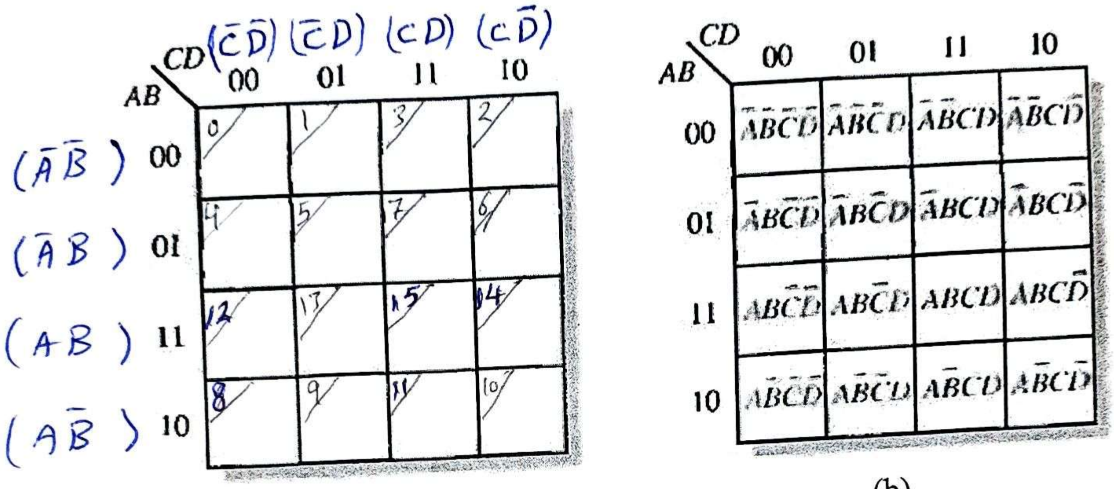


(a) (b)

Fig.(5-1) A 3-variable Karnaugh map showing product terms.

The 4-Variable Karnaugh Map

The 4-variable Karnaugh map is an array of sixteen cells, as shown in Fig.(5-2)(a). Binary values of A and B are along the left side and the values of C and D are across the top. The value of a given cell is the binary values of A and B at the left in the same row combined with the binary values of C and D at the top in the same column. For example, the cell in the upper right corner has a binary value of 0010 and the cell in the lower right corner has a binary value of 1010. Fig.(5-2)(b) shows the standard product terms that are represented by each cell in the 4-variable Karnaugh map.



(a) (b)

Fig.(5-2) A 4-variable Karnaugh map.

Cell Adjacency

The cells in a Karnaugh map are arranged so that there is only a single-variable change between adjacent cells. Adjacency is defined by a single-variable change. In the 3-variable map the 010 cell is adjacent to the 000 cell, the 011 cell, and the 110 cell. The 010 cell is not adjacent to the 001 cell, the 111 cell, the 100 cell, or the 101 cell.

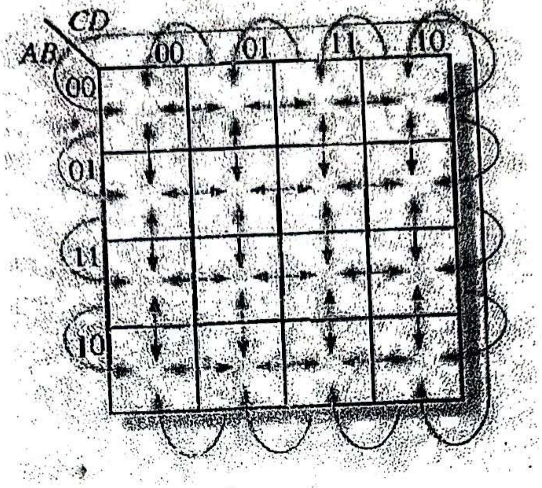
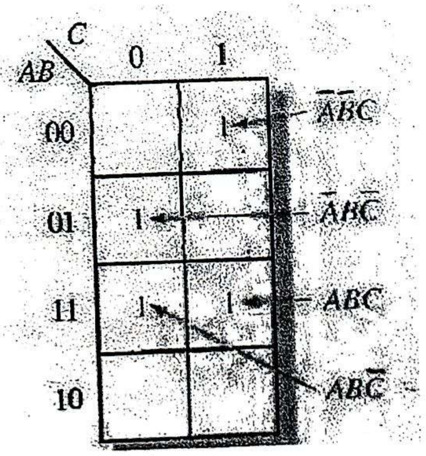


Fig.(5-3) Adjacent cells on a Karnaugh map are those that differ by only one variable. Arrows point between adjacent cells.

KARNAUGH MAP SOP MINIMIZATION

For an SOP expression in standard form, a 1 is placed on the Karnaugh map for each product term in the expression. Each 1 is placed in a cell corresponding to the value of a product term. For example, for the product term $A\bar{B}C$, a 1 goes in the 101 cell on a 3-variable map.

$$y = ABC + \bar{A}\bar{B}C + \bar{A}B\bar{C} + AB\bar{C}$$



Map the following standard SOP expression on a Karnaugh map:

$$y = \bar{A}\bar{B}CD + \bar{A}B\bar{C}\bar{D} + AB\bar{C}D + ABCD + AB\bar{C}\bar{D} + \bar{A}\bar{B}\bar{C}D + A\bar{B}C\bar{D}$$

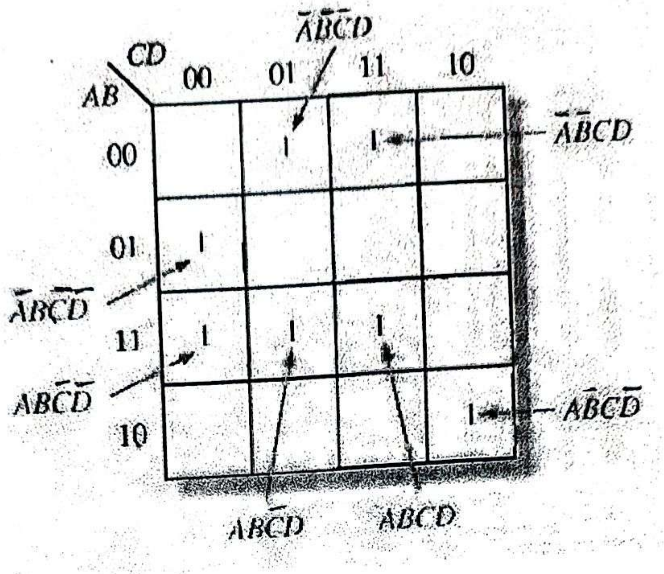


Fig.(5-5)

<u>Example</u>

Map the following SOP expression on a Karnaugh map: $\bar{A} + A\bar{B} + AB\bar{C}$.

<u>Solution</u>

The SOP expression is obviously not in standard form because each product term does not have three variables. The first term is missing two variables, the second term is missing one variable, and the third term is standard. First expand the terms numerically as follows:

$\overline{A}$	$+ A\overline{B}$	$+ AB\overline{C}$
000	100	110
001	101	
010		
011		

AB \ C	0	1
00	1	1
01	1	1
11	1	
10	1	1

Example

Map the following SOP expression on a Karnaugh map:

$$\overline{B}\,\overline{C} + A\overline{B} + AB\overline{C} + A\overline{B}C\overline{D} + \overline{A}\,\overline{B}\,\overline{C}D + A\overline{B}CD$$

Solution

The SOP expression is obviously not in standard form because each product term does not have four variables.

$\overline{B}\,\overline{C}$	$A\overline{B}$	$+ AB\overline{C}$	$+ A\overline{B}C\overline{D}$	$+ \overline{A}\,\overline{B}\,\overline{C}D$	$+ A\overline{B}CD$
0000	1000	1100	1010	0001	1011
0001	1001	1101			
1000	1010				
1001	1011				

Map each of the resulting binary values by placing a 1 in the appropriate cell of the 4- variable Karnaugh map.

AB \ CD	00	01	11	10
00	1	1		
01				
11	1	1		
10	1	1	1	1

Karnaugh Map Simplification of SOP Expressions

Grouping the 1s, you can group 1s on the Karnaugh map according to the following rules by enclosing those adjacent cells containing 1s. The goal is to maximize the size of the

groups and to minimize the number of groups.

- A group must contain either 1, 2, 4, 8, or 16 cells, which are all powers of two. In the case of a 3-variable map, $2^3 = 8$ cells is the maximum group.
- Each cell in a group must be adjacent to one or more cells in that same group.
- Always include the largest possible number of 1s in a group in accordance with rule 1.
- Each 1 on the map must be included in at least one group. The 1s already in a group can be included in another group as long as the overlapping groups include noncommon 1s.

Example:

Group the 1s in each of the Karnaugh maps in Fig.(5-6).

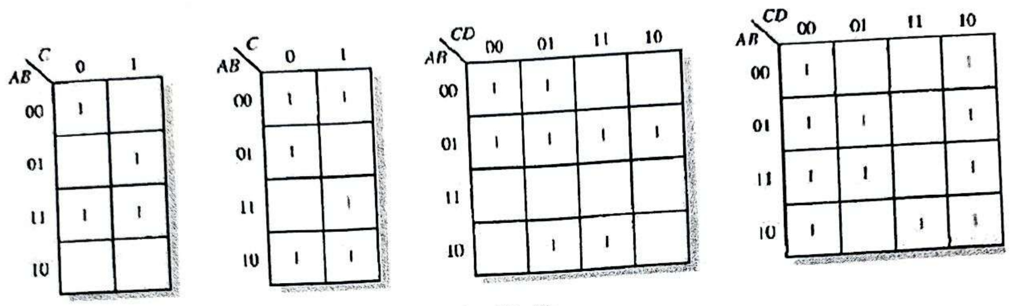


Fig.(5-6)

Solution:

The groupings are shown in Fig.(5-7). In some cases, there may be more than one way to group the 1s to form maximum groupings.

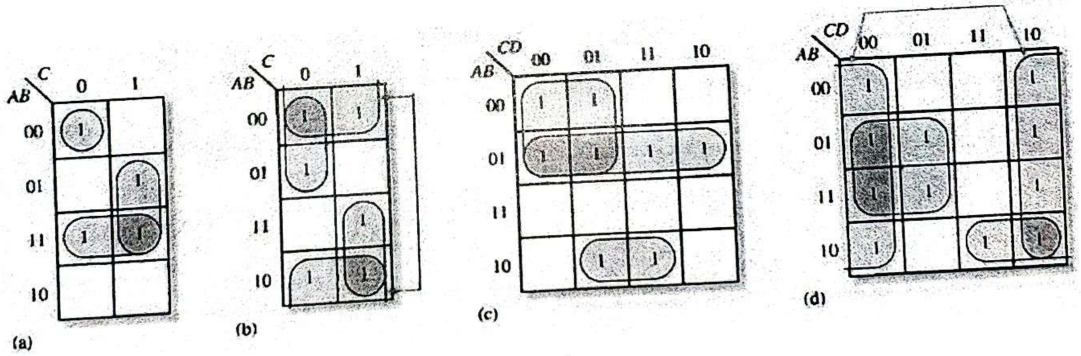


Fig.(5-7)

1)A 2-cell group eliminate one variables

2) A 4-cell group eliminate two variables

3) A 8-cell group eliminate three variables

4) A 16-cell group eliminate four variables

Example:

Determine the product terms for each of the Karnaugh maps in Fig.(5-8) and write the resulting minimum SOP expression.

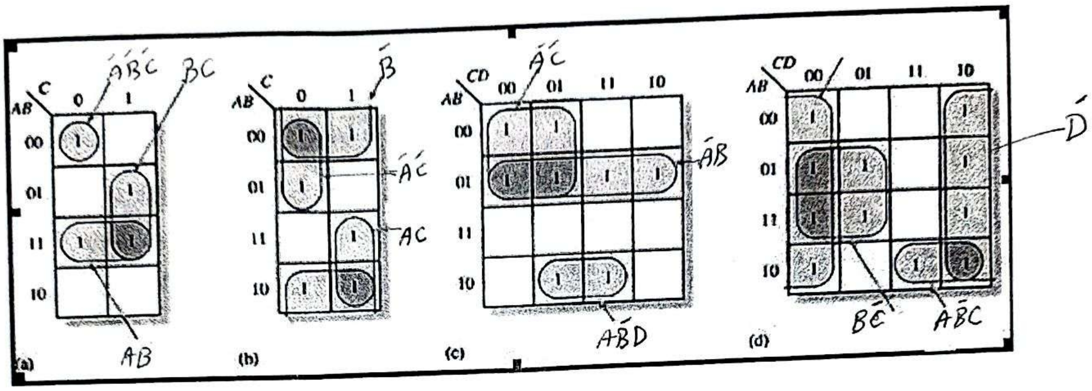


Fig.(5-8)

Solution:

The resulting minimum product term for each group is shown in Fig.(5-8). The minimum SOP expressions for each of the Karnaugh maps in the figure are:

(a) $AB+BC+\overline{A}\,\overline{B}\,\overline{C}$

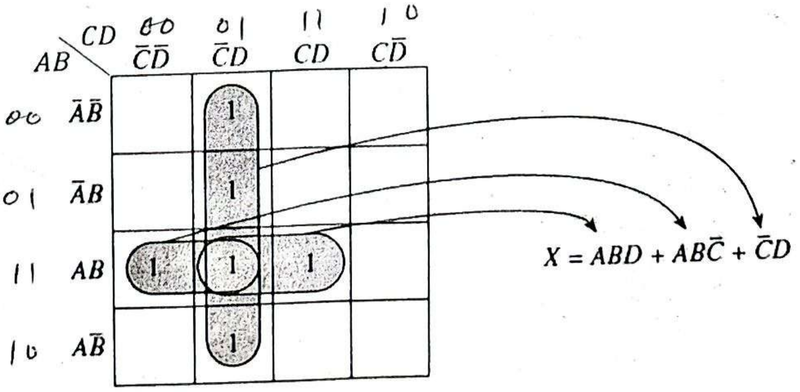
(b) $\overline{B} + AC + \overline{A}\,\overline{C}$

(C) $\overline{A}B + \overline{A}\,\overline{C} + A\overline{B}D$

(d) $\overline{D} + A\overline{B}C + B\overline{C}$

Example: Simplify the Boolean expression $X = \overline{A}B\overline{C}D + A\overline{B}\,\overline{C}D + \overline{A}\,\overline{B}\,\overline{C}D + AB\overline{C}D + AB\overline{C}\,\overline{D} + ABCD$ by using Karnaugh map?

Answer:



Karnaugh Map Simplification of POS Expressions

The process for minimizing a POS expression is basically the same as for an SOP expression except that you group 0s to produce minimum sum terms instead of grouping 1s to produce minimum product terms. The rules for grouping the 0s are the same as those for grouping the 1s that you learned before.

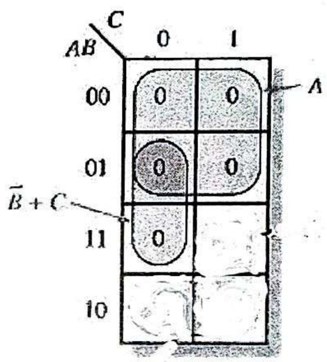
Example:

Use a Karnaugh map to minimize the following standard POS expression: Also, derive the equivalent SOP expression.

$$Y = (A + B + C)(A + B + \overline{C})(A + \overline{B} + C)(A + \overline{B} + \overline{C})(\overline{A} + \overline{B} + C)$$

000   001   010   011   110

Solution:



$\therefore Y = A\cdot(\overline{B}+C)$

Example:

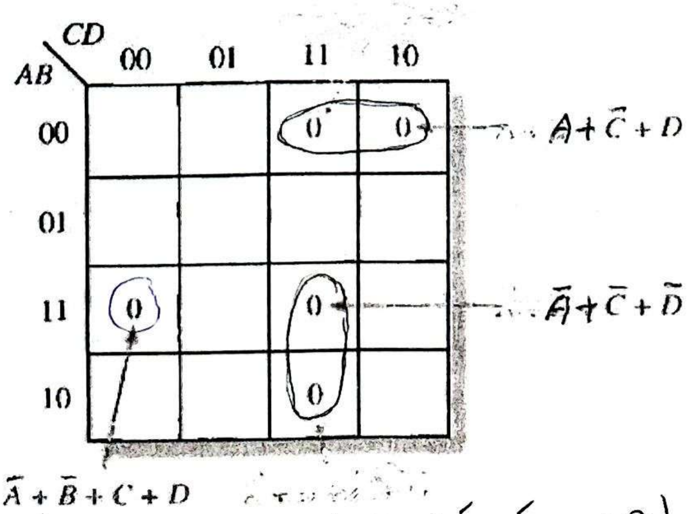
Map the following standard POS expression on a Karnaugh map:

$$Y = (\overline{A} + \overline{B} + C + D)(\overline{A} + B + \overline{C} + \overline{D})(A + B + \overline{C} + D)(\overline{A} + \overline{B} + \overline{C} + \overline{D})(A + B + \overline{C} + \overline{D})$$

Solution:

$$(\overline{A} + \overline{B} + C + D)(\overline{A} + B + \overline{C} + \overline{D})(A + B + \overline{C} + D)(\overline{A} + \overline{B} + \overline{C} + \overline{D})(A + B + \overline{C} + \overline{D})$$

1100   1011   0010   1111   0011

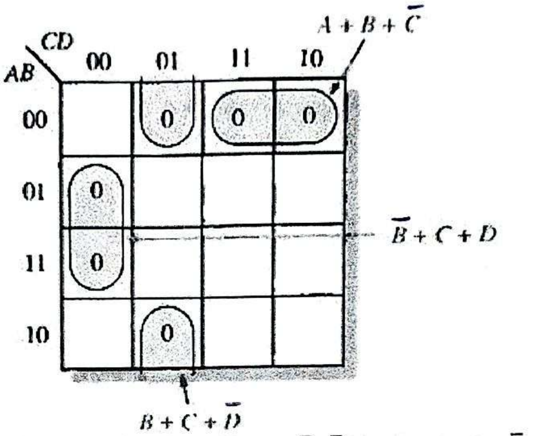


$\therefore Y = (A+\overline{C}+D)\cdot(\overline{A}+\overline{C}+\overline{D})\cdot(\overline{A}+\overline{B}+C+D)$

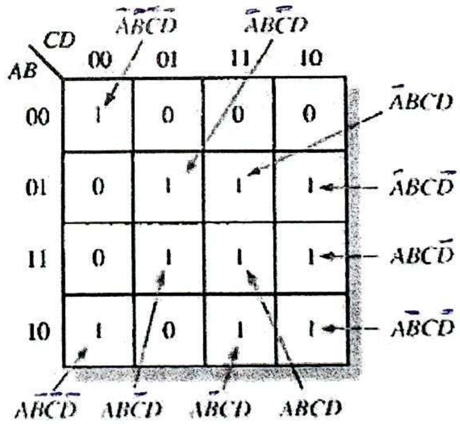
Ex: Simplify the following Boolean expression as S.O.P & P.O.S

minimum SOP expressions.

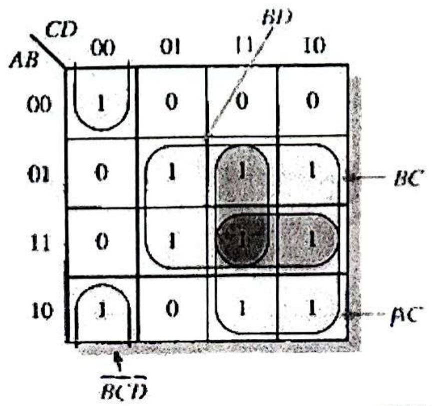
$$(\overline{A} + \overline{B} + C + D)(A + \overline{B} + C + D)(A + B + C + \overline{D})$$
$$(A + B + \overline{C} + \overline{D})(\overline{A} + B + C + \overline{D})(A + B + \overline{C} + D)$$



(a) Minimum POS: $(A + B + \overline{C})(\overline{B} + C + D)(B + C + \overline{D})$



(b) Standard SOP:



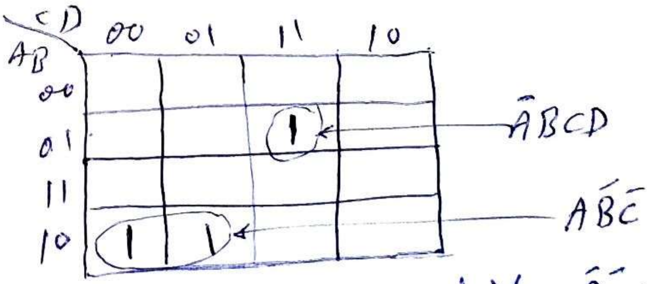
(c) Minimum SOP: $AC + BC + BD + \overline{B}\overline{C}\overline{D}$

"Don't Care" Conditions

Sometimes a situation arises in which some input variable combinations are not allowed. For example, recall that in the BCD code there are six invalid combinations: 1010, 1011, 1100, 1101, 1110, and 1111. Since these unallowed states will never occur in an application involving the BCD code, they can be treated as "don't care" terms with respect to their effect on the output. That is, for these "don't care" terms either a 1 or a 0 may be assigned to the output: it really does not matter since they will never occur.

The "don't care" terms can be used to advantage on the Karnaugh map. Fig.(5-9) shows that for each "don't care" term, an X is placed in the cell. When grouping the 1 s, the Xs can be treated as 1s to make a larger grouping or as 0s if they cannot be used to advantage. The larger a group, the simpler the resulting term will be.

The truth table in Fig.(5-9)(a) describes a logic function that has a 1 output only when the BCD code for 7,8, or 9 is present on the inputs. If the "don't cares" are used as 1s, the resulting expression for the function is A + BCD, as indicated in part (b). If the "don't cares" are not used as 1s, the resulting expression is $A\bar{B}\bar{C} + \bar{A}BCD$: so you can see the advantage of using "don't care" terms to get the simplest expression.

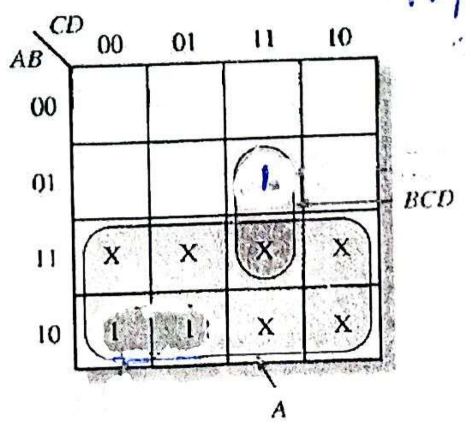


$\therefore Y = A\bar{B}\bar{C} + \bar{A}BCD$ without don't care

Inputs ABCD	Output Y
0000	0
0001	0
0010	0
0011	0
0100	0
0101	0
0110	0
0111	1
1000	1
1001	1
1010	X
1011	X
1100	X
1101	X
1110	X
1111	X

Don't cares

(a) Truth table



(b) Without "don't cares" $Y = A\bar{B}\bar{C} + \bar{A}BCD$
With "don't cares" $Y = A + BCD$

Fig.(5-9)

Example: Find a simplified logic expression for F = Σ (1, 3, 7, 11, 15) and d = Σ (0, 2, 5) using Karnaugh map? in S.O.P.

Decimal	A	B	C	D	Output(y)
0	0	0	0	0	X
1	0	0	0	1	1
2	0	0	1	0	X
3	0	0	1	1	1
5	0	1	0	1	X
7	0	1	1	1	1
11	1	0	1	1	1
15	1	1	1	1	1

AB \ CD	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$	X	1	1	X
$\bar{A}B$		X	1	
AB			1	
$A\bar{B}$			1	

$\bar{A}\bar{B}$

CD

∴ Output =

$y = \bar{A}\bar{B} + CD$

Ex: Simplify the function using k-map. $F = \sum m(0,2,5,8,10,12)$ & $d = \sum(1,4,6,7,15)$ in S.O.P

AB \ CD	00	01	11	10
00	1	X	0	1
01	X	1	X	X
11	1	0	X	0
10	1	0	0	1

$Y = \bar{B}\bar{D} + \bar{C}\bar{D} + \bar{A}B$

2. Exercises

1. Simplify the following logic expression using Karnough map: in S.O.P & P.O.S
    a. $X = A\overline{B}C + ABC + (C + D)(\overline{D} + E)$.
    b. $X = \overline{(A\overline{B} + \overline{A}B)}(A + B)$.
    c. $X = AB + \overline{A}CD + \overline{A}BD + \overline{A}C\overline{D} + ABCD$.
    d. $X = \overline{A}B + \overline{A}\,\overline{B}\,\overline{C} + A\,B\,\overline{C} + A\,\overline{B}\,\overline{C}$.

2. Find a simplified logic expression for Following using Karnaugh map: in S.O.P & P.O.S
    a. $F(A, B, C) = \Sigma(0, 1, 3)$.
    b. $F(A, B, C, D) = \Sigma(3\text{-}7, 9, 11, 12\text{-}15)$.
    c. $F = \Sigma(0, 1, 4, 5, 8, 9, 10, 11, 12)$

3. Find a simplified logic expression for Following using Karnaugh map in S.O.P & P.O.S
    a. $F = \Sigma(1, 3, 5, 7, 9)$ with $d = \Sigma(6, 12, 13)$.
    b. $F = \Sigma(4, 5, 7, 8, 10, 11, 13, 14)$ with $d = \Sigma(0, 1, 2)$.

4. Design logic circuits using k-map for each of the following:
    a. A logic circuit that gives 1 if the input is odd number, consider (0, 2) as don't care (use 3 digit as input).
    b. A logic circuit that gives 1 if the input >= 6 (use 4 digit as input).
    c. A logic circuit that gives 1 if the input equals (1,5,8), consider (0, 3 ,6) as don't care condition (use 4 digit as input).