No. 3

Standard and Canonical Forms

STANDARD FORMS OF BOOLEAN EXPRESSIONS

All Boolean expressions, regardless of their form, can be converted into either of two standard forms: the sum-of-products form or the product-of-sums form. Standardization makes the evaluation, simplification, and implementation of Boolean expressions much more systematic and easier.

1) *The Sum-of-Products (SOP) Form*

When two or more product terms are summed by Boolean addition, the resulting expression is a sum-of-products (SOP). Some examples are:

$AB + ABC$

$ABC + \overline{C}DE + \overline{B}C\overline{D}$

$AB + BCD + AC$

Also, an SOP expression can contain a single-variable term, as in

$A + AB\overline{C} + BC\overline{D}$.

In an SOP expression a single overbar cannot extend over more than one variable.

Example

Convert each of the following Boolean expressions to SOP form:

(a) $AB + B(CD + AC)$ $\Rightarrow$ $AB + BCD + ABC$

(b) $(A + B)(B + C + D)$ $\Rightarrow$ $AB + AC + AD + B + BC + BD$

(c) $\overline{(\overline{A + B}) + C}$ $\Rightarrow$ $(A+B)\cdot\overline{C} = A\overline{C} + B\overline{C}$

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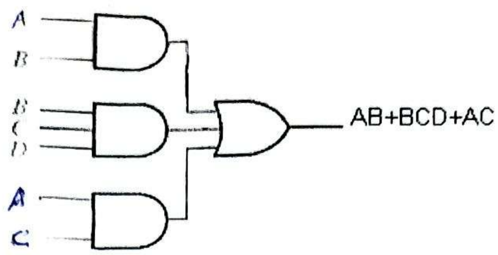
Ex: Impliment the expression $y = AB + AC + BCD$



Implementation of the SOP expression AB + BCD + AC.

Implementation of a SOP expression simply requires OR ing the outputs of two or more inputs AND gates.The sum terms is produced by on OR operation and the product terms is produced by an AND operation.Fig(4.18) shows the implementation of the expression AB+BCD+AC. The output X of the OR gate equals to the SOP expression.

The Standard SOP Form

So far, you have seen SOP expressions in which some of the product terms do not contain all of the variables in the domain of the expression. For example, the expression $\overline{A}B\overline{C} + A\overline{B}D + AB\overline{C}\overline{D}$ has a domain made up of the variables A, B, C. and D. However, notice that the complete set of variables in the domain is not represented in the first two terms of the expression; that is, D or $\overline{D}$ is missing from the first term and C or $\overline{C}$ is missing from the second term.

A standard SOP expression is one in which all the variables in the domain appear in each product term in the expression. For example, $\overline{A}BC\overline{D} + AB\overline{C}D + A\overline{B}CD$ is a standard SOP expression.

Converting Product Terms to Standard SOP:

Each product term in an SOP expression that does not contain all the variables in the domain can be expanded to standard SOP to include all variables in the domain and their complements. As stated in the following steps, a nonstandard SOP expression is converted into standard form using Boolean algebra rule 6 $(A + \bar{A} = 1)$ A variable added to its complement equals 1.

Step 1. Multiply each nonstandard product term by a term made up of the sum of a missing variable and its complement. This results in two product terms. As you know, you can multiply anything by 1 without changing its value.

Step 2. Repeat Step 1 until all resulting product terms contain all variables in the domain in either complemented or uncomplemented form. In converting a product term to standard form, the number of product terms is doubled for each missing variable.

Example

Convert the following Boolean expression into standard SOP form:

$$A\bar{B}C + \bar{A}\bar{B} + AB\bar{C}D$$

Solution

The domain of this SOP expression A, B, C, D. Take one term at a time. The first term, $A\bar{B}C$, is missing variable D or $\bar{D}$, so multiply the first term by $(D + \bar{D})$ as follows:

$$A\bar{B}C = A\bar{B}C(D + \bar{D}) = A\bar{B}CD + A\bar{B}C\bar{D}$$

In this case, two standard product terms are the result.

The second term, $\bar{A}\bar{B}$, is missing variables C or $\bar{C}$ and D or $\bar{D}$, so first multiply the second term by $C + \bar{C}$ as follows:

$$AB = \bar{A}\bar{B}(C + \bar{C}) = \bar{A}\bar{B}C + \bar{A}\bar{B}\bar{C}$$

The two resulting terms are missing variable D or $\overline{D}$, so multiply both terms by $(D + \overline{D})$ as follows:

$\overline{A}\overline{B}C(D + \overline{D}) + \overline{A}\overline{B}\overline{C}(D + \overline{D})$

$= \overline{A}\,\overline{B}CD + \overline{A}\overline{B}C\overline{D} + \overline{A}\overline{B}\overline{C}D + \overline{A}\overline{B}\overline{C}\overline{D}$

In this case, four standard product terms are the result.

The third term, $AB\overline{C}D$, is already in standard form. The complete standard SOP form of the original expression is as follows:

$A\overline{B}C + \overline{A}\overline{B} + AB\overline{C}D = A\overline{B}CD + A\overline{B}C\overline{D} + \overline{A}\,\overline{B}CD + \overline{A}\overline{B}C\overline{D} + \overline{A}\overline{B}\overline{C}D + \overline{A}\overline{B}\overline{C}\overline{D} + AB\overline{C}D$

2) *The Product-of-Sums (POS) Form*

A sum term was defined before as a term consisting of the sum (Boolean addition) of literals (variables or their complements). When two or more sum terms are multiplied, the resulting expression is a product-of-sums (POS). Some examples are

$(\overline{A} + B)(A + \overline{B} + C)$

$(A + \overline{B} + \overline{C})(C + \overline{D} + E)(B + C + D)$

$(A + \overline{B})(A + \overline{B} + C)(A + C)$

A POS expression can contain a single-variable term, as in

$A(A + B + C)(B + C + D)$.

In a POS expression, a single overbar cannot extend over more than one variable; however, more than one variable in a term can have an overbar. For example, a POS expression can have the term $\overline{A} + \overline{B} + \overline{C}$ but not $\overline{A + B + C}$.

Implementation of a POS Expression simply requires ANDing the outputs of two or more OR gates. A sum term is produced by an OR operation and the product of two or more sum terms is produced by an AND operation. Fig.(4-

20) shows for the expression (A + B)(B + C + D)(A + C). The output X of the AND gate equals the POS expression.

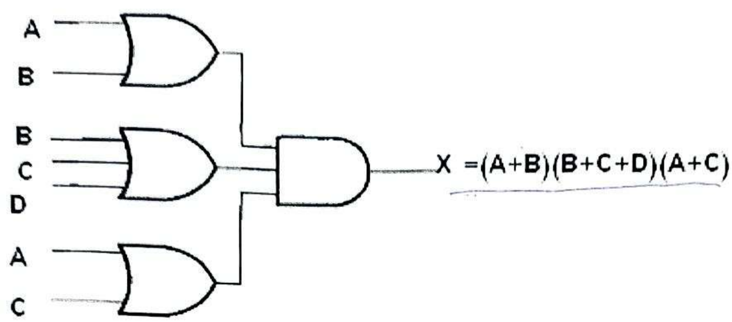


Fig.(4) Implementing of the P.O.S. expression

The Standard POS Form

So far, you have seen POS expressions in which some of the sum terms do not contain all of the variables in the domain of the expression. For example, the expression

$$(A + \overline{B} + C)\,(A + B + \overline{D})\,(A + \overline{B} + \overline{C} + D)$$

has a domain made up of the variables A, B, C, and D. Notice that the complete set of variables in the domain is not represented in e first two terms of the expression; that is, D or $\overline{D}$ is missing from the first term and C or $\overline{C}$ is missing from the second term.

A standard POS expression is one in which all the variables in the domain appear in each sum term in the expression. For example,

$(\overline{A} + \overline{B} + C + D)(A + \overline{B} + C + D)(A + B + C + D)$

is a standard POS expression. Any nonstandard POS expression (referred to simply as POS) can be converted to the standard form using Boolean algebra.

Converting a Sum Term to Standard POS

Each sum term in a POS expression that does not contain all the variables in the domain can be expanded to standard form to include all variables in the domain and their complements. As stated in the following steps, a nonstandard POS expression is converted into standard form using Boolean algebra rule 8 ($A\overline{A} = 0$) ...

Step 1. Add to each nonstandard product term a term made up of the product of the missing variable and its complement. This results in two sum terms. As you know, you can add 0 to anything without changing its value.

Step 2. Apply rule . $A + BC = (A + B)(A + C)$

Step 3. Repeat Step 1 until all resulting sum terms contain all variables in the domain in either complemented or noncomplemented form.

Example

Convert the following Boolean expression into standard POS form:

$$(\overline{A} + B + C)(\overline{B} + C + \overline{D})(A + \overline{B} + \overline{C} + D)$$

Solution

The domain of this POS expression is A, B, C, D. Take one term at a time. The first term, $\overline{A} + B + C$, is missing variable D or $\overline{D}$, so add $D\overline{D}$ and apply rule 12 as follows:

$$\overline{A} + B + C = \overline{A} + B + C + D\overline{D} = (\overline{A} + B + C + D)(\overline{A} + B + C + \overline{D})$$

The second term, $\overline{B} + C + \overline{D}$, is missing variable A or $\overline{A}$, so add $A\overline{A}$ and apply rule 12 as follows:

$$\overline{B} + C + \overline{D} = \overline{B} + C + \overline{D} + A\overline{A} = (A + \overline{B} + C + \overline{D})(\overline{A} + \overline{B} + C + \overline{D})$$

The third term, $A + B + \overline{C} + \overline{D}$, is already in standard form. The standard POS form of the original expression is as follows:

$$(\overline{A} + B + C)(\overline{B} + C + \overline{D})(A + \overline{B} + \overline{C} + D) =$$

$$(\overline{A} + B + C + D)(\overline{A} + B + C + \overline{D}) \cdot (A + \overline{B} + C + \overline{D})(\overline{A} + \overline{B} + C + \overline{D})(A + B + \overline{C} + \overline{D})$$

Examples:-

1. Identify each of the following expressions as SOP, standard SOP, POS, or standard POS:

   (a) $AB + \overline{A}BD + \overline{A}C\overline{D}$ (b) $(A + \overline{B} + C)(A + B + \overline{C})$

   (c) $\overline{A}BC + AB\overline{C}$ (d) $A(A + \overline{C})(A + B)$

2. Convert each SOP expression in Question 1 to standard form.
3. Convert each POS expression in Question 1 to standard form.

CANONICAL FORMS OF BOOLEAN EXPRESSIONS

With one variable x & $\overline{x}$.

With two variables $\overline{x}\,\overline{y}$, $x\,\overline{y}$, $\overline{x}\,y$ and $x\,y$.

With three variables $\overline{x}\,\overline{y}\,\overline{z}$, $\overline{x}\,\overline{y}\,z$, $\overline{x}\,y\,\overline{z}$, $\overline{x}\,y\,z$, $x\,\overline{y}\,\overline{z}$, $x\,\overline{y}\,z$, $x\,y\,\overline{z}$ & $x\,y\,z$.

These eight AND terms are called minterms.

n variables can be combined to form 2^n minterms.

x	y	z	minterm	designation	maxterm	designation
0	0	0	$\overline{x}\,\overline{y}\,\overline{z}$	m_0	$x+y+z$	M_0
0	0	1	$\overline{x}\,\overline{y}\,z$	m_1	$x+y+\overline{z}$	M_1
0	1	0	$\overline{x}\,y\,\overline{z}$	m_2	$x+\overline{y}+z$	M_2
0	1	1	$\overline{x}\,y\,z$	m_3	$x+\overline{y}+\overline{z}$	M_3
1	0	0	$x\,\overline{y}\,\overline{z}$	m_4	$\overline{x}+y+z$	M_4
1	0	1	$x\,\overline{y}\,z$	m_5	$\overline{x}+y+\overline{z}$	M_5
1	1	0	$x\,y\,\overline{z}$	m_6	$\overline{x}+\overline{y}+z$	M_6
1	1	1	$x\,y\,z$	m_7	$\overline{x}+\overline{y}+\overline{z}$	M_7
			(AND terms)		(OR terms)	

Note that each maxterm is the complement of its corresponding minterm and vice versa.

For example the function F

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

$F = \bar{x}\,\bar{y}\,z + x\,\bar{y}\,\bar{z} + x\,y\,z$

$F = m_1 \quad + m_4 + m_7$

$\therefore F = \sum m(1, 4, 7)$

Any Boolean function can be expressed as a sum of minterms (sum of products **SOP**) or product of maxterms (product of sums **POS**).

$\bar{F} = \bar{x}\,\bar{y}\,\bar{z} + \bar{x}\,y\,\bar{z} + \bar{x}\,y\,z + x\,\bar{y}\,z + x\,y\,\bar{z}$

The complement of $\bar{F} = \bar{\bar{F}} = F$

$F = (x + y + z)\,(x + \bar{y} + z)\,(x + \bar{y} + \bar{z})\,(\bar{x} + y + \bar{z})\,(\bar{x} + \bar{y} + z)$

$F = M_0\,M_2\,M_3\,M_5\,M_6$

$\therefore F = \prod M(0, 2, 3, 5, 6)$

<u>Example</u>

Express the Boolean function $F = A + \bar{B}C$ in a sum of minterms (SOP).

<u>Solution</u>

The term A is missing two variables because the domain of F is (A, B, C)

$A = A(B + \bar{B}).(C + \bar{C})$ because $B + \bar{B} = 1$

$= ABC + AB\bar{C} + A\bar{B}C + A\bar{B}\bar{C}$

$\overline{B}C$ missing A, so

$$\overline{B}C(A + \overline{A}) = A\overline{B}C + \overline{A}\overline{B}C$$

$$\therefore F = ABC + AB\overline{C} + \underline{A\overline{B}C} + A\overline{B}\overline{C} + \underline{A\overline{B}C} + \overline{A}\overline{B}C$$

$$\boxed{\text{Because } A + A = A}$$

$$F = ABC + AB\overline{C} + A\overline{B}C + A\overline{B}\overline{C} + \overline{A}\overline{B}C$$

$$F = m_7 + m_6 + m_5 + m_4 + m_1$$

In short notation

$$F(A, B, C) = \sum(1, 4, 5, 6, 7)$$

To convert from one canonical form to another, interchange the symbols $\sum$, $\prod$ and list those numbers missing from the original form.

$$F = M_4\, M_5\, M_0\, M_2 = m_1 + m_3 + m_6 + m_7$$

$$F(x, y, z) = \prod(0, 2, 4, 5) = \sum(1, 3, 6, 7)$$