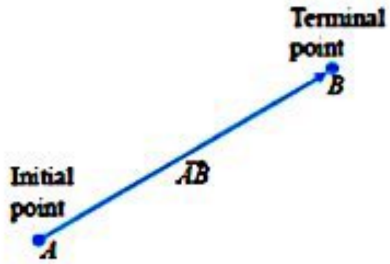
Vector Analysis

Scalar and Vector

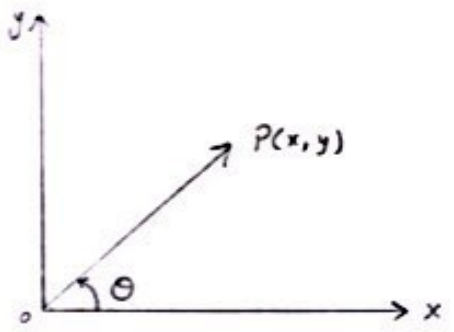
Scalar: is a quantity like volume, temperature and time, which have **magnitude only**. The scalar is number.

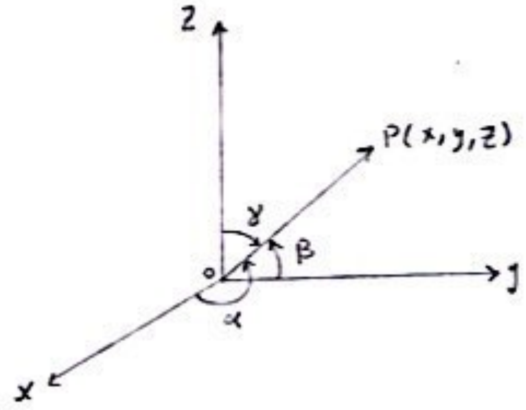
Vector: is a quantity like velocity, acceleration and force, which have **magnitude and direction.**

The vector is represented by a directed line segment (arrow) $\overrightarrow{AB}$ from one point A called the initial point to another point B called the terminal point, as show in figure below.



The vector $\vec{A}$ can be represented in plane and in the space as follows:



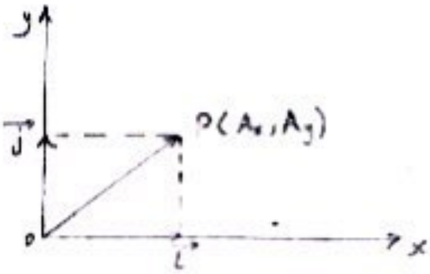


θ determines the direction of $\vec{A}$ in the plane (2D).

α, β, γ determine the direction of $\vec{A}$ in the space (3D).

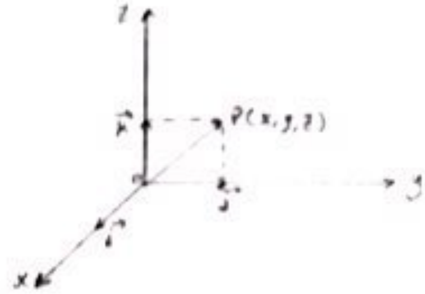
The **magnitude or length of the vector** is denoted by $|\vec{A}|$ is a scalar quantity. In the xy plane the components vector can be written a $\vec{A} = A_x\ \vec{\imath} + A_y\ \vec{\jmath}$, the magnitude of $\vec{A}$ is:

$$|\vec{A}| = \sqrt{(A_x)^2 + (A_y)^2}$$



In the space, the components vector can be written as $\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}$, the magnitude of $\vec{A}$ is:

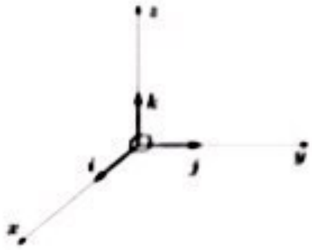
$$|\vec{A}| = \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2}$$



Where $\vec{\imath}, \vec{\jmath}$ *and* $\vec{k}$ are unit vectors having the direction of the positive x, y and z axes of a rectangular coordinate system.

$|\vec{\imath}| = |\vec{\jmath}| = |\vec{k}| = 1$

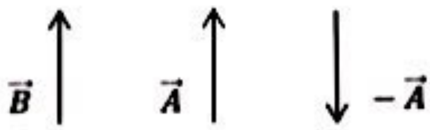
and $\vec{\imath} \perp \vec{\jmath},\ \vec{\imath} \perp \vec{k},\ \vec{\jmath} \perp \vec{k}$



Note:

- If the vector having zero length is called **Zero vector** and denoted by $\vec{O}$.
- If two vectors $\vec{A}$ and $\vec{B}$ have the same magnitude and direction regardless of their initial points called **Equal vector**, thus $\vec{A} = \vec{B}$. For example,

  $\vec{A} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}$ and $\vec{B} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k} \Rightarrow \vec{A} = \vec{B}$.
- If two vectors have the same magnitude but opposite direction, such vector called the **Inverse of vector** denoted by $-\vec{A}$. For example, $\vec{A} = -\vec{A}$.



$|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Ex: $A = (-1, 3, 2)$, find $|\vec{A}|$?

$|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$= \sqrt{(-1)^2 + (3)^2 + (2)^2}$

$= \underline{\underline{\sqrt{14}}}$ → Scalar ناتج يكون

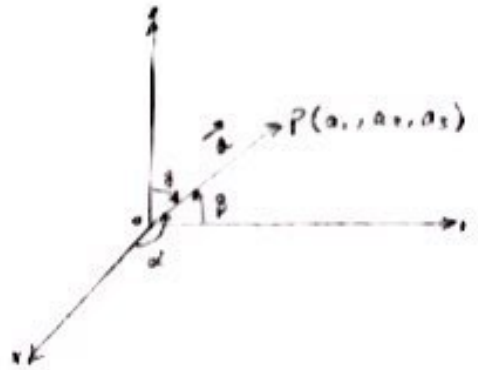
② $B = (2, -7, 0)$, find $|\vec{B}|$?

$|\vec{B}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

$= \sqrt{(2)^2 + (-7)^2 + (0)^2} = \sqrt{53}$

Direction Cosines of a Vector

The cosines of α, β and γ angles are called direction cosines of the vector $\vec{a} = a_1\ \vec{i} + a_2\,\vec{j} + a_3\,\vec{k}$



$$\cos\alpha = \frac{a_1}{|\vec{a}|}\ , \qquad \cos\beta = \frac{a_2}{|\vec{a}|}\ , \quad \cos\gamma = \frac{a_3}{|\vec{a}|}$$

Note: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Example:

Find the direction cosines of the vector $\vec{a} = 2\vec{i} - 3\,\vec{j} + 2\,\vec{k}$?

Sol.

$$|\vec{a}| = \sqrt{(2)^2 + (-3)^2 + (2)^2} = \sqrt{4+9+4} = \sqrt{17}$$

$$\cos\alpha = \frac{a_1}{|\vec{a}|} = \frac{2}{\sqrt{17}}\ , \qquad \cos\beta = \frac{a_2}{|\vec{a}|} = \frac{-3}{\sqrt{17}}\ , \qquad \cos\gamma = \frac{a_3}{|\vec{a}|} = \frac{2}{\sqrt{17}}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\left(\frac{2}{\sqrt{17}}\right)^2 + \left(\frac{-3}{\sqrt{17}}\right)^2 + \left(\frac{2}{\sqrt{17}}\right)^2 = \frac{4}{17} + \frac{9}{17} + \frac{4}{17} = \frac{17}{17} = 1$$

Unit Vector

The vector $\vec{A}$ having a unit vector $\vec{u}$ in the same direction of it and given by

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|}$$

Example:

Find a unit vector in the same direction of $\vec{A} = \vec{i} - 2\vec{j} + 3\vec{k}$?

Sol.

$$|\vec{A}| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{1+4+9} = \sqrt{14}$$

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{i} - 2\vec{j} + 3\vec{k}}{\sqrt{14}} = \frac{\vec{i}}{\sqrt{14}} - \frac{2\vec{j}}{\sqrt{14}} + \frac{3\vec{k}}{\sqrt{14}}$$

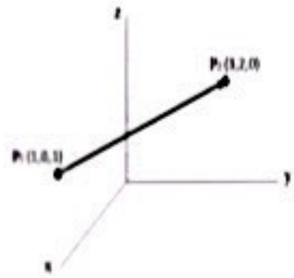
Example:

Find a unit vector in the direction of the vector from P_1 (1,0,1) to P_2 (3,2,0)?

Sol.

Vector between two points ($\overrightarrow{P_1P_2}$) represents the vector:

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$



Then

$$\overrightarrow{P_1P_2} = (3-1)\vec{i} + (2-0)\vec{j} + (0-1)\vec{k} = 2\vec{i} + 2\vec{j} - \vec{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\vec{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{3} = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{1}{3}\vec{k}$$

Example:

Find a unit vector in the direction of the vector from P(5,7, -1) to Q(2,9, -2)?

Sol.

$$\overrightarrow{PQ} = (2-5)\vec{i} + (9-7)\vec{j} + (-2-(-1))\vec{k} = -3\vec{i} + 2\vec{j} - \vec{k}$$

$$|\overrightarrow{PQ}| = \sqrt{(-3)^2 + (2)^2 + (-1)^2} = \sqrt{9+4+1} = \sqrt{14}$$

$$\vec{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{-3\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{14}}$$

العمليات الحسابية على المتجهات

1) الجمع

$A = (a_1, a_2, a_3)$
$B = (b_1, b_2, b_3)$

* $A+B = (a_1+b_1 , a_2+b_2 , a_3+b_3)$.

Ex: 1) $A = (3, 4, 1)$
$B = (2, 1, -1)$
Find $A+B$?

$A = 3i + 4j + 1k$.
$B = 2i + 1j - 1k$

$A+B = (a_1+b_1 , a_2+b_2 , a_3+b_3)$

$= (3+2, 4+1, 1+(-1))$
$= (5, 5, 0)$
$\therefore A+B = 5i + 5j + 0k$.

2) $A = (2, -7, 0)$
$B = (1, -3, -5)$

Find $A+B$?

② الطرح

$* \; A-B = (a_1-b_1, a_2-b_2, a_3-b_3)$.

Ex: $A = (3,4,1)$

$B = (2,1,-1)$

Find $A-B$? $A = 3i+4j+1k$

$B = 2i+1j-1k$.

$A-B = (a_1-b_1, a_2-b_2, a_3-b_3)$.

$A-B = (3-2, 4-1, 1-(-1))$

$A-B = (1, 5, 2)$

$\therefore A-B = 1i+5j+2k$.

③ ضرب المتجه بعدد صحيح.

* عند ضرب المتجه بعدد صحيح فيضرب المتجه بالعدد.

Ex: ① $A = (3,4,1)$ find $2A$.

$2A = 2(3,4,1) = (6,8,2)$

② $B = (6,0,-7)$ find $3B$.

$3B = 3(6,0,-7) = (18,0,-21)$.

Example:

Let $\vec{a} = -\vec{i} + 3\vec{j} + \vec{k}$ and $\vec{b} = 4\,\vec{i} + 7\vec{j}$, find $2\vec{a} + 3\vec{b}$, $\vec{a} - \vec{b}$, $\left|\frac{1}{2}\,\vec{a}\right|$?

Sol.

$$2\vec{a} + 3\vec{b} = 2\left(-\vec{i} + 3\vec{j} + \vec{k}\right) + 3(4\,\vec{i} + 7\vec{j})$$

$$= \left(-2\,\vec{i} + 6\,\vec{j} + 2\,\vec{k}\right) + (12\,\vec{i} + 21\,\vec{j}) = 10\,\vec{i} + 27\,\vec{j} + 2\,\vec{k}$$

$$\vec{a} - \vec{b} = \left(-\vec{i} + 3\vec{j} + \vec{k}\right) - (4\,\vec{i} + 7\vec{j}) = -5\vec{i} - 4\vec{j} + \vec{k}$$

$$\frac{1}{2}\,\vec{a} = \frac{1}{2}\left(-\vec{i} + 3\vec{j} + \vec{k}\right) = -\,\frac{1}{2}\,\vec{i} + \frac{3}{2}\vec{j} + \frac{1}{2}\vec{k}$$

$$\left|\frac{1}{2}\,\vec{a}\right| = \sqrt{\left(-\,\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{9}{4} + \frac{1}{4}} = \sqrt{\frac{11}{4}}$$

① Dot product → Scalar. ناتجه يكون

$A \cdot B = a_1b_1 + a_2b_2 + a_3b_3$.

Ex: $A = (2, -2, 1)$

$B = (5, 8, 1)$, find $A \cdot B$.

$\vec{A} \cdot \vec{B} = a_1b_1 + a_2b_2 + a_3b_3$

$= (2 * 5) + (-2 * 8) + (1 * 1)$

$= 10 - 16 + 1 = \underline{\underline{-5}}$

② Cross product: vector. ناتجه يكون

$$\vec{A} \times \vec{B} = (\overset{i}{a_2b_3 - a_3b_2}, \overset{j}{a_3b_1 - a_1b_3}, \overset{K}{a_1b_2 - a_2b_1}).$$

$A = (2, -2, 1)$, $B = (5, 8, 1)$

$\vec{A} \times \vec{B} = ((-2 \times 1) - (1 \times 8), (1 \times 5) - (2 \times 1), (2 \times 8) - (-2 \times 5))$

$\vec{A} \times \vec{B} = (-10i, 3j, 26k)$

طريقة المحددات (3)

$$\vec{A} = a_1 i + a_2 j + a_3 k$$

$$\vec{B} = b_1 i + b_2 j + b_3 k.$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{A} \times \vec{B} = + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i = (a_2 \times b_2) - (a_3 b_2)$$

ونفس الطريقة بالنسبة لباقي المحددات.

Ex:

$\vec{u} = i - 2j + k$

$\vec{v} = 3i + j - 2k$, find $\vec{u} \times \vec{v}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = + \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} i - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} j + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} k$$

$$\vec{u} \times \vec{v} = (4-1)i - (-2-3)j + (-1+6)k$$

$$= 3i + 5j + 7k$$

H.W find $\vec{v} \times \vec{u}$

Example:

If $\vec{A} = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{B} = -4\,\vec{i} + 3\,\vec{j} + \vec{k}$, find $\vec{A} \times \vec{B}$, $\vec{B} \times \vec{A}$ and $\vec{A} \times \vec{A}$?

Sol.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \vec{k}$$

$$= [(1 * 1) - (1 * 3)]\,\vec{i} - [(2 * 1) - (1 * -4)]\,\vec{j} + [(2 * 3) - (1 * -4)]\vec{k}$$

$$= -2\vec{i} - 6\vec{j} + 10\,\vec{k}$$

$$\vec{B} \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} -4 & 1 \\ 2 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} -4 & 3 \\ 2 & 1 \end{vmatrix} \vec{k}$$

$$= [(3 * 1) - (1 * 1)]\,\vec{i} - [(-4 * 1) - (1 * 2)]\,\vec{j} + [(-4 * 1) - (3 * 2)]\vec{k}$$

$$= 2\vec{i} + 6\vec{j} - 10\,\vec{k}$$

$\therefore\ \vec{A} \times \vec{B} = -\,(\vec{B} \times \vec{A})$

$$\vec{A} \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \vec{k}$$

$$= [(1 * 1) - (1 * 1)]\,\vec{i} - [(2 * 1) - (1 * 2)]\,\vec{j} + [(2 * 1) - (1 * 2)]\vec{k}$$

$$= 0$$

projection vector المسقط الاتجاهي

* the projection of $\vec{B}$ onto $\vec{A}$

$$\underset{A}{\text{proj}}\,\vec{B} = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|^2}\,\vec{A}$$

* the projection of $\vec{A}$ onto $\vec{B}$

$$\underset{B}{\text{proj}}\,\vec{A} = \frac{\vec{A}\cdot\vec{B}}{|\vec{B}|^2}\,\vec{B}$$

Ex: if $B = (2, 1, -1)$, $A = (1, 0, 2)$

find the projection $\vec{B}$ onto $\vec{A}$

sol

$$\underset{A}{\text{proj}}\,\vec{B} = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|^2}\,\vec{A}$$

Scanned with CamScanner

$$|\vec{A}| = \sqrt{1^2 + 0^2 + (-2)^2}$$

$$= \sqrt{5}$$

$$|\vec{A}|^2 = \sqrt{5} \times \sqrt{5} = 5$$

$$\vec{A} \cdot \vec{B} = 2\times1 + 1\times0 + (-1)\times(-2)$$

$$= 2 + 0 + 2 = 4.$$

$$\vec{A} = 1i + 0j - 2K$$

$$\text{proj}_{A} \vec{B} = \frac{4}{5}(i + 0j - 2K)$$

$$= \frac{4}{5}(i - 2K) = \frac{4}{5}i - \frac{8}{5}K.$$

<u>H.w</u> find the projection $\vec{A}$ onto $\vec{B}$.

Example:

If $\vec{A} = \vec{i} + 2\,\vec{j} + \vec{k}$ and $\vec{B} = 4\vec{i} - 4\,\vec{j} + 7\,\vec{k}$, find the vector projection of $\vec{A}$ onto $\vec{B}$, and also $\vec{B}$ onto $\vec{A}$?

Sol.

$$proj_{\vec{B}}\,\vec{A} = \left(\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|^2}\right)\,\vec{B}$$

$$= \frac{(1*4)+(2*-4)+(1*7)}{\left(\sqrt{(4)^2+(-4)^2+(7)^2}\right)^2}\;(4\vec{i} - 4\,\vec{j} + 7\,\vec{k})$$

$$= \frac{4-8+7}{16+16+49}\,(4\vec{i} - 4\,\vec{j} + 7\,\vec{k})$$

$$= \frac{3}{81}\,(4\vec{i} - 4\,\vec{j} + 7\,\vec{k}) = \frac{12}{81}\vec{i} - \frac{12}{81}\,\vec{j} + \frac{21}{81}\,\vec{k}$$

$$proj_{\vec{A}}\,\vec{B} = \left(\frac{\vec{B}\cdot\vec{A}}{|\vec{A}|^2}\right)\,\vec{A}$$

$$= \frac{3}{\left(\sqrt{(1)^2+(2)^2+(1)^2}\right)^2}\;(\vec{i} + 2\,\vec{j} + \vec{k})$$

$$= \frac{3}{6}\,(\vec{i} + 2\,\vec{j} + \vec{k}) \quad = \frac{1}{2}\vec{i} + \vec{j} + \frac{1}{2}\,\vec{k}$$

- parallel and orthogonal vectors:

- vectors are orthogonal if:

$\vec{A} \cdot \vec{B} = 0.$

- vectors are parallel if:

(a) $\vec{A} \times \vec{B} = 0$

or

(b) $\vec{A} = C\vec{B}.$

Ex: Determite whether the given vectors are orthogonal, parallel or neither..

- $\vec{u} = (-2, 6, -4)$. $\vec{V} = (4, -12, 8).$

sol

$\vec{V} = 2\vec{u}$, so the vectors are parallel.

- $\vec{u} \cdot \vec{V} = (-2 * 4) + (6 * -12) + (-4 * 8)$

$= -8 - 32 - 32$

$= -72.$

So the vectors are not orthogonal.

Parametric Equations

(x_0, y_0, z_0) point

(a, b, c) vector

$x = x_0 + at$

$y = y_0 + bt$

$z = z_0 + ct$

point vector

Ex: find the parametric equation for point $(5, 1, 3)$ which is paralled to vector $(i + 4j - 2k)$.

sol

$x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$

$x = 5 + 1t$, $y = 1 + 4t$, $z = 3 - 2t$.

Vector Equation

$r = r_0 + t v$

r_0: النقطة المعطاة (x_0, y_0, z_0)

v: متجه موازي (a, b, c).

Ex: Find the vector equation to the line that pass through the point $(5,1,3)$ and parallel to vector

$\vec{V} = i + 4j - 2K$.

Sol: $r_0 = 5i + 1j + 3K$.

$\vec{V} = i + 4j - 2K$

$r = r_0 + t v$

$r = 5i + j + 3K + t(\vec{i + 4j - 2K})$

$r = 5i + j + 3K + ti + 4tj - 2tK$.

$r = (5+t)i + (1+4tj) + (3-2t)K$.

Planes Equation

vectors

$$\vec{a}(x-x_0)+\vec{b}(y-y_0)+\vec{c}(z-z_0)=0.$$

Ex: Find the plane equation for point (5,3,4) and vector (1,2,7).

$$\begin{array}{ccc} 1, & 2, & 7 \\ \downarrow & \downarrow & \downarrow \\ a & b & c \end{array}$$

Sol

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$1(x-5)+2(y-3)+7(z-4)=0$$

$$x-5+2y-6+7z-28=0$$

$$x+2y+7z-39=0$$

$$x+2y+7z=39.$$