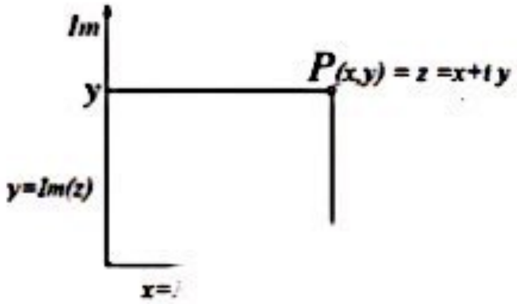
Complex Numbers

A complex number is a number of the form $z = x + i\,y$, where x and y are real numbers and i is imaginary unit$(i = \sqrt{-1})$ with the property that $i^2 = -1$. also complex number can be written as the ordered pair (x,y) and plotted as a point in plane called the complex plane.



In complex plane: the horizontal x-axis called the real axis **(Re)** and the vertical y-axis called the im...

Remarks:

1- The set all complex number will be denoted by $\mathbb{C}$.

2- Electrical engineers use j instead of i because i reserved for the current.

3- Let z_1 and $z_2 \in \mathbb{C}$, then $z_1 = z_2$ if $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

Example: Solve the equation $2x^2 + 3x + 5 = 0$.

Sol.

$$x = \frac{-3 \mp \sqrt{(3)^2 - (4 * 2 * 5)}}{2 * 2}$$

$$= \frac{-3 \mp \sqrt{9 - 40}}{4}$$

$$= \frac{-3 \mp \sqrt{-31}}{4} = \frac{-3 \mp i\,\sqrt{31}}{4}$$

The modulus and The conjugate of Complex Numbers

The **modulus** of complex number $z = x + \iota\, y$ is $\boxed{|z| = \sqrt{x^2 + y^2}\,.}$

The **conjugate** of complex number is obtained by changing the sign of the imaginary part. Hence the complex conjugate of $z = x + \iota\, y$ is $\boxed{\bar{z} = x - \iota\, y.}$

Properties of Modulus and Conjugate of Complex Numbers:

Let **z** be a complex number then

1) $|z| = |-z| = |\bar{z}|$.

2) $|z|^2 = |z^2| = z\,\bar{z} = (Re(z))^2 + (Im(z))^2$

3) $\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|$ for all $z_1, z_2 \in \mathbf{C}$.

4) $|z_1 z_2| = |z_1||z_2|$ for all $z_1, z_2 \in \mathbf{C}$.

5) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ for all $z_1, z_2 \in \mathbf{C}$.

6) $|z_1 z_2|^2 = |z_1|^2 |z_2|^2 = (|z_1|\,|z_2|)^2$ for all $z_1, z_2 \in \mathbf{C}$.

7) $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$ for all $z_1, z_2 \in \mathbf{C}$.

8) $\overline{z_1\, z_2} = \bar{z_1} \cdot \bar{z_2}$ for all $z_1, z_2 \in \mathbf{C}$.

9) $\bar{\bar{z}} = z$

10) $z \cdot \bar{z} = |z|^2$

Operations on Complex Numbers

1) Addition and Subtraction of Complex Numbers

Two complex numbers are added/subtracted by added/subtracted separately the two real parts and the two imaginary parts.

If $z_1 = a + i\,b$ and $z_2 = c + i\,d$

Then $\quad z_1 + z_2 = (a + c) + i(b + d)$

$\quad\quad\quad z_1 - z_2 = (a - c) + i(b - d)$

Example:

If $z_1 = 2 + 4\,i$ *and* $z_2 = 3 - i$, *find* $z_1 + z_2$, $z_1 - z_2$ *and* $z_2 - z_1$?

Sol.

$z_1 + z_2 = (2 + 3) + i(4 - 1) = 5 + i\,3$

$z_1 - z_2 = (2 - 3) + i\big(4 - (-1)\big) = -1 + i\,5$

$z_2 - z_1 = (3 - 2) + i(-1 - 4) = 1 - i\,5$

2) Multiplication and Division of Complex Numbers

If $z_1 = a + i\,b$ and $z_2 = c + i\,d$ two complex numbers, then:

The multiplication of two complex numbers z_1 and z_2 is defined by:

$z_1 z_2 = (a + i\,b)(c + i\,d) = ac + i\,ad + i\,bc + i^2\,bd$

since $i^2 = -1$

$$\therefore \boxed{z_1 z_2 = (ac - bd) + i(ad + bc)}$$

<u>The division</u> of two complex numbers z_1 and z_2 is defined by:

$$\frac{z_1}{z_2} = \frac{a + i\,b}{c + i\,d}$$

$$= \frac{a + i\,b}{c + i\,d} * \frac{c - i\,d}{c - i\,d} = \frac{ac - i\,ad + i\,bc - i^2 bd}{c^2 + d^2}$$

$$= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$$

$$\therefore \quad \boxed{\frac{z_1}{z_2} = \frac{(ac + bd)}{c^2 + d^2} + i\,\frac{(bc - ad)}{c^2 + d^2}}$$

<u>Example:</u>

If $z_1 = 1 - 3\,i$, $z_2 = -2 + 5\,i$ and $z_3 = -3 - 4\,i$, find:

a) $z_1 z_2$ b) $\frac{z_1}{z_3}$ c) $\frac{z_1\,z_2}{z_1 + z_2}$ d) $z_1\,z_2\,z_3$

<u>Sol</u>.

a) $z_1 z_2 = (1 - 3\,i)\,(-2 + 5\,i) = -2 + 5\,i + 6\,i + 15 = 13 + 11\,i$

b) $\frac{z_1}{z_3} = \frac{(1-3\,i)}{(-3-4\,i)} * \frac{(-3+4\,i)}{(-3+4\,i)} = \frac{-3+4i+9i+12}{9+16} = \frac{9+13\,i}{25}$

c) $\frac{z_1\,z_2}{z_1 + z_2}$ **(H.W)** **<u>Ans.</u>** $\frac{9}{5} - i\,\frac{37}{5}$

d) $z_1\,z_2\,z_3$ **(H.W)** **<u>Ans.</u>** $5 - i\,85$

Complex equations

If two complex numbers are equal, then their real parts are equal and their imaginary parts are equal. Hence if $a + i\,b = c + i\,d$, then $a = c$ and $b = d$.

Example: Solve the complex equations:

a) $2(x + iy) = 6 - 3\,i$

b) $(1 + 2\,i)(-2 - 3\,i) = a + i\,b$

c) $(2 - 3\,i) = \sqrt{a + i\,b}$ (H.W) **Ans.** $a = -5\,, b = -12$

d) $(x - i\,2y) + (y - i\,3x) = 2 + 3\,i$ **Ans.** $x = -7\,,\ y = 9$

Sol.

a) $2(x + i\,y) = 6 - 3\,i$

$2x + i\,2y = 6 - 3\,i$

$\therefore\ 2x = 6 \ \Rightarrow\ x = 3$

$\therefore\ 2y = -3 \ \Rightarrow\ y = \dfrac{-3}{2}$

b) $(1 + 2\,i)(-2 - 3\,i) = a + i\,b$

$-2 - 3\,i - 4\,i + 6 = a + i\,b$

$4 - 7i = a + i\,b$

$\therefore\ a = 4 \ ,\ b = -7$