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Second Class / Mathematics II

Eigenvalues and Eigenvectors

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Eigenvalues and Eigenvectors

- The vector **x** is an eigenvector of matrix ***A*** and λ is an eigenvalue of *A* if: $Ax = \lambda x$ (assume non-zero x)
- Eigenvalues and eigenvectors are only defined for square matrices (i.e., $m = n$)
- Eigenvectors are not unique (e.g., if λ is an eigenvector, so is $k\lambda$)
- Zero vector is a trivial solution to the eigenvalue equation for any number λ and is not considered as an eigenvector.
- **Interpretation:** the linear transformation implied by *A* cannot change the direction of the eigenvectors λ, but change only their magnitude.

We summarize the computational approach for determining eigenpairs (λ, **x**) (eigenvalues and eigen vector) as a two-step procedure:

Step I. To find the eigenvalues of **A** compute the roots of the characteristic equation $\det(\mathbf{A} - \lambda \mathbf{I}_n) = 0$.

Step II. To find an eigenvector corresponding to an eigenvalue μ, compute a nontrivial solution to the homogeneous linear system $(\mathbf{A} - \mu \mathbf{I}_n)\mathbf{x} = \mathbf{0}$.

Example1: Find eigenpairs of $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

Step I. Find the eigenvalues. $c(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}_2) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix}\right)$

$$= (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6$$

Thus the characteristic polynomial is a quadratic and the eigenvalues are the solutions of $\lambda^2 - 5\lambda + 6 = 0$. We factor the quadratic to get $(\lambda - 3)(\lambda - 2) = 0$ so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$.

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Step II. To find corresponding **$(A - \lambda_i I_n)x = 0$** for

Case $\lambda_1 = 3$: We have that **$(A - 3I_2)x = 0$** has augmented matrix

$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ -2 & 1 & 0 \end{array}\right]$ and its rref is $\left[\begin{array}{cc|c} 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$. (Verify.) Thus if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we have

$x_1 = (1/2)x_2$ so $\mathbf{x} = \begin{bmatrix} (1/2)x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$, $x_2 \neq 0$. Choosing $x_2 = 2$, to conveniently get integer entries, gives eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Case $\lambda_2 = 2$: We have that **$(A - 2I_2)p = 0$** has augmented matrix

$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array}\right]$ and its rref is $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$. (Verify.) Thus if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we have

$x_1 = x_2$ so $\mathbf{x} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x_2 \neq 0$. Choosing $x_2 = 1$ gives eigenvector

$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Note: rref means row reduced echelon form.

Computing λ and v

- To find the eigenvalues λ of a matrix *A*, find the roots of the *characteristic polynomial* :

$$det(A - \lambda I) = 0$$

Example 2: $A = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix}$

Ax= λx

$$det(\begin{bmatrix} 5-\lambda & -2 \\ 6 & -2-\lambda \end{bmatrix}) = 0 \text{ or } \lambda^2 - 3\lambda + 2 = 0 \text{ or } \lambda_1 = 1,\ \lambda_2 = 2$$

$$x_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Example 3:

For our matrix $A = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix}$ we write the characteristic equation:

$$\underbrace{\begin{vmatrix} 2-\lambda & 0 & 3 \\ 1 & 2-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{vmatrix}}_{\text{The determinant}} = \underbrace{(2-\lambda)^3 + 3[-3(2-\lambda)]}_{\text{The expansion}} = \underbrace{-\lambda^3 + 6\lambda^2 - 3\lambda - 10}_{\text{The standard form}} = \underbrace{(2-\lambda)(\lambda-5)(\lambda+1)}_{\text{The factorization}} = 0$$

for $\lambda_1 = 2$

$$\begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{k}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

for $\lambda_2 = 5$

$$\begin{pmatrix} -3 & 0 & 3 \\ 1 & -3 & 0 \\ 3 & 0 & -3 \end{pmatrix} \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{k}_2 = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$$

for $\lambda_3 = -1$

$$\begin{pmatrix} 3 & 0 & 3 \\ 1 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix} \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{k}_3 = \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}$$

Example 4: Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

The characteristic polynomial is $\lambda^3 - 6\lambda^2 + 11\lambda - 6$

Its factors are $(\lambda - 1)(\lambda - 2)(\lambda - 3)$

So the eigenvalues are 1, 2, and 3.

Corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$