Al-Rasheed University Collage

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First Class / Mathematics

The Definite Integral

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THEOREM 7—Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Proof Let F denote any antiderivative of f. Then,

$$\begin{aligned}
\int_a^b f(g(x)) \cdot g'(x)\, dx &= F(g(x)) \Big]_{x=a}^{x=b} \\
&= F(g(b)) - F(g(a)) \\
&= F(u) \Big]_{u=g(a)}^{u=g(b)} \\
&= \int_{g(a)}^{g(b)} f(u)\, du.
\end{aligned}$$

$\frac{d}{dx} F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$

Fundamental Theorem, Part 2 ■

To use the formula, make the same u-substitution $u = g(x)$ and $du = g'(x)\, dx$ you would use to evaluate the corresponding indefinite integral. Then integrate the transformed integral with respect to u from the value $g(a)$ (the value of u at $x = a$) to the value $g(b)$ (the value of u at $x = b$).

EXAMPLE 1 Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

Solution We have two choices.

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 7.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \int_{0}^{2} \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

Evaluate the new definite integral.

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du \qquad \text{Let } u = x^3 + 1,\ du = 3x^2\,dx.$$

$$= \frac{2}{3}u^{3/2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{2}{3}(x^3+1)^{3/2} + C \qquad \text{Replace } u \text{ by } x^3 + 1.$$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1} \qquad \text{Use the integral just found, with limits of integration for } x.$$

$$= \frac{2}{3}\left[((1)^3+1)^{3/2} - ((-1)^3+1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3} \qquad \blacksquare$$

EXAMPLE 2 We use the method of transforming the limits of integration.

(a) $$\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta \, d\theta = \int_1^0 u \cdot (-du)$$

Let $u = \cot\theta,\ du = -\csc^2\theta\, d\theta,$
$-du = \csc^2\theta\, d\theta.$
When $\theta = \pi/4,\ u = \cot(\pi/4) = 1.$
When $\theta = \pi/2,\ u = \cot(\pi/2) = 0.$

$$= -\int_1^0 u\, du$$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

(b) $$\int_{-\pi/4}^{\pi/4} \tan x\, dx = \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x}\, dx$$

$$= -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u}$$

Let $u = \cos x,\ du = -\sin x\, dx.$
When $x = -\pi/4,\ u = \sqrt{2}/2.$
When $x = \pi/4,\ u = \sqrt{2}/2.$

$$= -\ln|u|\Big]_{\sqrt{2}/2}^{\sqrt{2}/2} = 0$$

Integrate, zero width interval