Al-Rasheed University Collage
Dept. of Medical Instrument Tech. Eng.
First Class / Mathematics

Method of integration (Integration by Parts)

Roweda.M.Mohammed

$$\frac{d}{dx}\left[f\left(x\right)\cdot g\left(x\right)\right]=\frac{d}{dx}[f(x)]g(x)+\frac{d}{dx}[g(x)]f(x)$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$\int u\,dv = uv - \int v\,du.$$

To use this technique we need to identify likely candidates for $u = f(x)$ and $dv = g'(x)\,dx$.

EXAMPLE 1 Find

$$\int x \cos x \, dx.$$

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x \, dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

There are four apparent choices available for u and dv in Example 1:

1. Let $u = 1$ and $dv = x \cos x \, dx$.
2. Let $u = x$ and $dv = \cos x \, dx$.
3. Let $u = x \cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x \, dx$.

EXAMPLE 2 Find

$$\int \ln x \, dx.$$

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \frac{1}{x} dx, \qquad v = x.$ Simplest antiderivative

Then from Equation (2),

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C.$$ ■

Sometimes we have to use integration by parts more than once.

EXAMPLE 3 Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $u = x^2$, $dv = e^x \, dx$, $du = 2x \, dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x \, dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

Using this last evaluation, we then obtain

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2 \int x e^x \, dx \\ &= x^2 e^x - 2x e^x + 2e^x + C, \end{aligned}$$

Use this technique for :

$$\int x^n e^x \, dx$$

EXAMPLE 7 Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

EXAMPLE 4 Evaluate

$$\int e^x \cos x\, dx.$$

Solution Let $u = e^x$ and $dv = \cos x\, dx$. Then $du = e^x\, dx$, $v = \sin x$, and

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x\, dx, \qquad v = -\cos x, \qquad du = e^x\, dx.$$

Then

$$\int e^x \cos x\, dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x\, dx)\right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx.$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x\, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

■

EXAMPLE 5 Obtain a formula that expresses the integral

$$\int \cos^n x\, dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x\, dx,$$

so that

$$du = (n-1)\cos^{n-2} x\,(-\sin x\, dx) \qquad \text{and} \qquad v = \sin x.$$

Integration by parts then gives

$$\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx.$$

If we add

$$(n-1)\int \cos^n x\, dx$$

to both sides of this equation, we obtain

$$n\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$$

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in Example 5 tells us that

$$\int \cos^3 x\, dx = \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\, dx$$

$$= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.$$

EXAMPLE 8.4.1 Evaluate $\int x \ln x\,dx$. Let $u = \ln x$ so $du = 1/x\,dx$. Then we must let $dv = x\,dx$ so $v = x^2/2$ and

$$\int x \ln x\,dx = \frac{x^2 \ln x}{2} - \int \frac{x^2}{2}\frac{1}{x}\,dx = \frac{x^2 \ln x}{2} - \int \frac{x}{2}\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.$$

□

EXAMPLE 8.4.2 Evaluate $\int x \sin x\,dx$. Let $u = x$ so $du = dx$. Then we must let $dv = \sin x\,dx$ so $v = -\cos x$ and

$$\int x \sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

□

EXAMPLE 8.4.3 Evaluate $\int \sec^3 x\,dx$. Of course we already know the answer to this, but we needed to be clever to discover it. Here we'll use the new technique to discover the antiderivative. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x \tan x\,dx$ and $v = \tan x$ and

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int \tan^2 x \sec x\,dx\\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x\,dx\\ &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx.\end{aligned}$$

At first this looks useless—we're right back to $\int \sec^3 x\, dx$. But looking more closely:

$$\begin{aligned}
\int \sec^3 x\, dx &= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx \\
\int \sec^3 x\, dx + \int \sec^3 x\, dx &= \sec x \tan x + \int \sec x\, dx \\
2\int \sec^3 x\, dx &= \sec x \tan x + \int \sec x\, dx \\
\int \sec^3 x\, dx &= \frac{\sec x \tan x}{2} + \frac{1}{2}\int \sec x\, dx \\
&= \frac{\sec x \tan x}{2} + \frac{\ln|\sec x + \tan x|}{2} + C.
\end{aligned}$$

EXAMPLE 8.4.4 Evaluate $\int x^2 \sin x\,dx$. Let $u = x^2$, $dv = \sin x\,dx$; then $du = 2x\,dx$ and $v = -\cos x$. Now $\int x^2 \sin x\,dx = -x^2 \cos x + \int 2x \cos x\,dx$. This is better than the original integral, but we need to do integration by parts again. Let $u = 2x$, $dv = \cos x\,dx$; then $du = 2$ and $v = \sin x$, and

$$\begin{aligned}
\int x^2 \sin x\,dx &= -x^2 \cos x + \int 2x \cos x\,dx \\
&= -x^2 \cos x + 2x \sin x - \int 2 \sin x\,dx \\
&= -x^2 \cos x + 2x \sin x + 2 \cos x + C.
\end{aligned}$$

□

Exercises

Find the antiderivatives.

1. $\int x \cos x \, dx \Rightarrow$

2. $\int x^2 \cos x \, dx \Rightarrow$

3. $\int x e^x \, dx \Rightarrow$

4. $\int x e^{x^2} \, dx \Rightarrow$

5. $\int \sin^2 x \, dx \Rightarrow$

6. $\int \ln x \, dx \Rightarrow$

7. $\int x \arctan x \, dx \Rightarrow$

8. $\int x^3 \sin x \, dx \Rightarrow$

9. $\int x^3 \cos x \, dx \Rightarrow$

10. $\int x \sin^2 x \, dx \Rightarrow$

11. $\int x \sin x \cos x \, dx \Rightarrow$

12. $\int \arctan(\sqrt{x}) \, dx \Rightarrow$

13. $\int \sin(\sqrt{x}) \, dx \Rightarrow$

14. $\int \sec^2 x \csc^2 x \, dx \Rightarrow$