Al-Rasheed University Collage

Dept. of Medical Instrument Tech. Eng.

First Class / Mathematics

Techniques of Integration:

Trigonometric Integrals

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$$\cos^2\theta + \sin^2\theta = 1, \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta.$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$1 + \tan^2\theta = \sec^2\theta.$$

$$1 + \cot^2\theta = \csc^2\theta.$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

TABLE 8.1 Basic integration formulas

1. $$\int k\,dx = kx + C \qquad \text{(any number } k\text{)}$$
2. $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$
3. $$\int \frac{dx}{x} = \ln|x| + C$$
4. $$\int e^x\,dx = e^x + C$$
5. $$\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$$
6. $$\int \sin x\,dx = -\cos x + C$$
7. $$\int \cos x\,dx = \sin x + C$$
8. $$\int \sec^2 x\,dx = \tan x + C$$
9. $$\int \csc^2 x\,dx = -\cot x + C$$
10. $$\int \sec x \tan x\,dx = \sec x + C$$
11. $$\int \csc x \cot x\,dx = -\csc x + C$$
12. $$\int \tan x\,dx = \ln|\sec x| + C$$
13. $$\int \cot x\,dx = \ln|\sin x| + C$$
14. $$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$
15. $$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$
16. $$\int \sinh x\,dx = \cosh x + C$$
17. $$\int \cosh x\,dx = \sinh x + C$$
18. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$
19. $$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$
20. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$
21. $$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \qquad (a > 0)$$
22. $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad (x > a > 0)$$

EXAMPLE 1 Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

Solution This is an example of Case 1.

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx \qquad m \text{ is odd.}$$

$$= \int (1 - \cos^2 x)(\cos^2 x)(-d(\cos x)) \qquad \sin x \, dx = -d(\cos x)$$

$$= \int (1 - u^2)(u^2)(-du) \qquad u = \cos x$$

$$= \int (u^4 - u^2) \, du \qquad \text{Multiply terms.}$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

EXAMPLE 2 Evaluate

$$\int \cos^5 x \, dx.$$

Solution This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 \, d(\sin x) \qquad \cos x \, dx = d(\sin x)$$

$$= \int (1 - u^2)^2 \, du \qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4) \, du \qquad \text{Square } 1 - u^2.$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

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$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

EXAMPLE 3 Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Solution This is an example of Case 3.

$$\begin{aligned}\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\, dx \\ &= \frac{1}{8}\int (1+\cos 2x-\cos^2 2x-\cos^3 2x)\, dx \\ &= \frac{1}{8}\left[x+\frac{1}{2}\sin 2x-\int(\cos^2 2x+\cos^3 2x)\, dx\right]\end{aligned}$$

m and *n* both even

For the term involving $\cos^2 2x$, we use

$$\begin{aligned}\int \cos^2 2x \, dx &= \frac{1}{2}\int (1+\cos 4x)\, dx \\ &= \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right).\end{aligned}$$

Omitting the constant of integration until the final result

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x \, dx = \int (1 - \sin^2 2x) \cos 2x \, dx$$

$u = \sin 2x,$
$du = 2 \cos 2x \, dx$

$$= \frac{1}{2} \int (1 - u^2) \, du = \frac{1}{2} \left(\sin 2x - \frac{1}{3} \sin^3 2x \right).$$

Again omitting C

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16} \left(x - \frac{1}{4} \sin 4x + \frac{1}{3} \sin^3 2x \right) + C.$$

EXAMPLE 4 Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

Solution To eliminate the square root, we use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2 \theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Therefore,

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx \qquad \text{$\cos 2x \geq 0$ on $[0, \pi/4]$}$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1 - 0] = \frac{\sqrt{2}}{2}.$$

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EXAMPLE 5 Evaluate

$$\int \tan^4 x\, dx.$$

Solution

$$\int \tan^4 x\, dx = \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x \cdot (\sec^2 x - 1)\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x\, dx$$

and have

$$\int u^2\, du = \frac{1}{3}u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C.$$

EXAMPLE 6 Evaluate

$$\int \sec^3 x\, dx.$$

Solution We integrate by parts using

$$u = \sec x, \qquad dv = \sec^2 x\, dx, \qquad v = \tan x, \qquad du = \sec x \tan x\, dx.$$

Then

$$\begin{aligned}\int \sec^3 x\, dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x\, dx) \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx \qquad \tan^2 x = \sec^2 x - 1 \\ &= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx.\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

and

$$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln \left|\sec x + \tan x\right| + C.$$ ■

EXAMPLE 7 Evaluate

$$\int \tan^4 x \sec^4 x \, dx.$$

Solution

$$\int (\tan^4 x)(\sec^4 x)\, dx = \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x)\, dx \qquad \sec^2 x = 1 + \tan^2 x$$

$$= \int (\tan^4 x + \tan^6 x)(\sec^2 x)\, dx$$

$$= \int (\tan^4 x)(\sec^2 x)\, dx + \int (\tan^6 x)(\sec^2 x)\, dx$$

$$= \int u^4\, du + \int u^6\, du = \frac{u^5}{5} + \frac{u^7}{7} + C \qquad u = \tan x,\ du = \sec^2 x\, dx$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$$

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$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x],$$
$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x],$$
$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x].$$

EXAMPLE 8 Evaluate

$$\int \sin 3x \cos 5x\, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned}\int \sin 3x \cos 5x\, dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x]\, dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$